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Nonlinear Dispersion Model of Sea Waves in the Coastal Zone
沿岸域에서의 非線形 波浪 分散模型

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Abstract□The problem of sea wave transformation in the coastal zone taking into account effects of nonlinearity and disperison has been studied. Mathematical model for description of regular wave transformation is based on the method of nonlinear ray theory. The equations for rays and wave field have been produced. Nonlinear wave field is described by the modified Korteweg-de Vries equation. Some analytical solutions of this equation are obtained. Caustic transformation and dissipation efffects are included in the mathematical model. Numerical algorithm of solution of the Korteweg-de Vries equation and its stability criterion are described. Results of nonlinear transformation of sea waves in the coastal zone are demonstrated.

要 旨: 파랑의 비선형성 및 분산을 고려한, 연안역에서의 파랑변형에 관한 연구를 수행하였다. 규칙파의 변형에 관한 수학적 모형은 비선형 ray모델에 기초하였으며, ray 및 파동장에 관한 방정식들을 수립하였다. 비선형 파동장은 수정 Korteweg-de Vries 식으로서 나타내었으며, 이에 대한 몇몇 해석해들을 구하였다. 또한 Caustic 변형 및 감쇄효과를 수학적 모형에 포함하였다. Korteweg-de Vries 방정식에 대한 수치계산 알고리즘과 안정조건을 기술하였으며, 연안역에서의 비선형 파랑변형 계산결과를 제시하였다.

1. INTRODUCTION

Sea wave propagation in the coastal zone is affected by various factors. First, bottom relief that leads to refraction of wave beams and fronts. The related problems have received due to attention in a linear approximation for both regular and irregular waves (Davidan *et al.*, 1985; Mei, 1989; Massel, 1990; Dingemans, 1993). Second, wave dispersion results in strong wave intermittence due to the difference between phase and group velocities; it is often taken into account for waves in the coastal zone using the well-known Berkhoff model or its modifications (Berkhoff, 1976; Booij, 1981; Kozlov and Pelinovsky, 1989; Mei, 1989; Massel, 1990; Madsen *et al.*, 1991; Dingemans, 1993). Third, dissipation in a bottom boundary layer and in the region of breaking which is usually taken into consideration by means of various empirical relations (Davidan *et al.*, 1985; Mei, 1989; Leontjev, 1989; Massel, 1990). Finally, an important factor is nonlinearity that changes the spectral composition of waves and facilitates wave breaking. The effect of nonlinearity on water waves was considered, in its most pure form, by Stoker (1957) and Whitham (1974) who, generally, omitted the specific marine aspects of the problem.

Linear transformation of sea waves in the coastal zone, except a relatively narrow run-up region, is usually calculated within an energy balance equation that is valid for both regular and irregular waves (Davidan *et al.*, 1985; Mei, 1989; Massel, 1990; Nekrasov and Pelinovsky, 1992). In the first case, in the absence of dissipation, this equation reduces to the conservation of energy flux along the ray

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beam, while in the second case, to the conservation of spatial spectrum of waves along the beam. With nonlinearity taken into account, statistical independence of spectral wave components is no longer valid and is connected with strong synchronization of harmonics in shallow water. Substantial changes in the wave profile, including its breaking, is a good illustration of effects of synchronization. Consequently, nonlinear sea waves are considered to be regular as a rule.

This paper is concerned with analytical and numerical analysis of regular-wave transformation taking into account nonlinearity and dispersion in the coastal zone using nonlinear ray method developed by Shen (1975) and Ostrovsky and Pelinovsky (1975). This method enables one to obtain known relations for the construction of rays-trajectories of the wave packet motion-in the framework of a linear theory of shallow water, as well as a nonlinear evolution Korteweg-de Vries eqution for the wave field propagating along "linear" rays. Of particular interest is the description of nonlinear field at caustics and at focal points. It will be shown that linear models are quite sufficient for the description of caustic transformation of a nonlinear field. Test examples demonstrating the variability of sea waves in the coastal zone are presented.

2. NONLINEAR RAY METHOD

Basic equations for nonlinear dispersion water wave theory may be derived in different ways: employing asymptotic methods, Galerkin procedures, etc. (Stoker, 1957; Whitham, 1974; Engelbrecht *et al.*, 1988; Mei, 1989). Analysis is, as a rule, confined to first-order expansion terms with respect to dispersion and nonlinearity, and the corresponding equations are written in the form (Peregrine, 1967):

$$\frac{\partial\vec{u}}{\partial t}+(\vec{u}\nabla)\vec{u}+g\nabla\eta=\vec{G}=\frac{1}{h}\nabla\left(\frac{h^3}{3}\nabla\frac{\partial\vec{u}}{\partial t}-\frac{h^2}{2}\nabla h\frac{\partial\vec{u}}{\partial t}\right)-\nabla h\left(\frac{h}{2}\nabla\frac{\partial\vec{u}}{\partial t}\right)+\nabla h\frac{\partial\vec{u}}{\partial t},$$

$$\frac{\partial\eta}{\partial t}+\mathrm{div}[(h+\eta)\vec{u}]=0 \qquad (1)$$

Note that such equations may be written for wave processes of arbitrary amplitude, given small dispersion, (Su and Gardner, 1969; Zheleznyak and Pelinovsky, 1985) which seems to be promising for the description of run-up zone too. Since the wave amplitude is small when the waves are present far from shore line, we can restrict ourselves to the system of Eq. (1). It is convenient to pass from system (1) over to a single wave equation for water level in linear (left-hand) side:

$$\frac{\partial^2\eta}{\partial t^2}-\mathrm{div}[c^2\nabla\eta]=Q$$
$$=-\mathrm{div}(\eta\vec{u})+\mathrm{div}[h(\vec{u}\nabla)\vec{u}]-\mathrm{div}(h\vec{G}), \qquad (2)$$

where $c=\sqrt{gh}$, and $h(x,y)$ is an unperturbed depth of the basin. Assuming $|\nabla h|\ll 1$ and analyzing the waves moving only in one direction, we can introduce instead of time t the variable $s=\tau(\vec{r})-t$, with τ determined below in the text. In the new variables, Eq. (2) will take on a form

$$[1-c^2(\nabla\tau)^2]\frac{\partial^2\eta}{\partial s^2}-\frac{\partial}{\partial s}[2c^2\nabla\tau\nabla\eta+\eta\,\mathrm{div}(c^2\nabla\tau)]$$
$$-\mathrm{div}(c^2\nabla\eta)=Q \qquad (3)$$

So as to determine unambiguously all the terms of this equation we will employ some physical assumptions. Assume that the curvature radius of the wave front is large (a quasiplane wave approximation) and the depth varies slowly (mild bottom slope). Then it is natural to regard that the solution depends primarily on one coordinate, s, and $\vec{r}$-dependence is weaker. Consequently, the terms containing second-order derivatives and the squares of first-order derivatives with respect to the slow coordinate r may be neglected in Eq. (3) in the first approximation. Then Eq. (3) can be written as two independent equations:

$$(\nabla\tau)^2=c^{-2}=(gh)^{-1}, \qquad (4)$$

$$\frac{\partial}{\partial s}(2\nabla\tau\nabla\eta+\eta\nabla\tau+\eta\nabla\tau\nabla c^2/c^2)+Q\{\eta,\ \vec{u}\}=0 \qquad (5)$$

Eq. (4) is well known in the linear theory of long waves. It is an eikonal equation that allows for the calculation of wave trajectories(rays). It should be emphasized that equation contains only local ocean depth and hence in this approximation the ray picture does not depend on wave dispersion and nonlinearity. The procedure for the calculation of rays is well known. Because Eq. (4) is nonlinear partial differential equation of first order, it can be treated in general way by using the method of characteristics. Let $y(x)$ represent a particular ray, corresponding equation for it can be obtained from Eq. (4) in the following form (see, for example, Mei (1989) and Dingemans (1993)):

$$\frac{d}{dx}\left(\frac{c^{-1}\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}\right)=\sqrt{1+\left(\frac{dy}{dx}\right)^2}\frac{\partial c^{-1}}{\partial y}, \qquad (6)$$

as a conclusion of the Fermat's principle. Now many numerous computer programs for the calculation of rays are available.

Therfore, we will omit them and pass over directly to Eq. (5). Because nonlinearity is small we can express flow velocity in the right-hand side of Eq. (5) by means of formulas from a linear theory of long waves: $\vec{u}=\vec{l}\eta/c$ where $\vec{l}$ is a unit vector in the propagation direction, along a ray. Using rays as a reference coordinate system, after a series of simple transformations used in linear ray theory, Eq. (5) may be written in the form

$$\sqrt{gh}\frac{\partial\eta}{\partial l}+\frac{3\eta}{2h}\frac{\partial\eta}{\partial s}+\frac{h}{6g}\frac{\partial^3\eta}{\partial s^3}+\frac{\sqrt{gh}}{4h\Delta^2}\frac{d(h\Delta^2)}{dl}\eta=0 \qquad (7)$$

Here l is the distance along the ray, Δ is the ray separation factor(the differential width of the ray beam), which is the distance between two neighboring rays, and s is the time in the co-moving coording using availble ray programs. Eq. (7) is a modification of the Korteweg-de Vries equation and transforms into the latter at a constant depth of the basin and a plane wave front. Eq. (7) was obtained by Ostrovsky and Pelinovsky (1970) and Johnson (1972) for one-dimensional case and by basic equation for the description of evolution of tsunami waves in the coastal zone (Pelinovsky, 1982; Voltzinger *et al.*, 1989). Its application to wind waves needs some modifications and adoptation which will be discussed below.

It should be noted that, with nonlinearity and dispersion neglected, Eq. (7) yields the relation of energy flux conservation

$$\eta^2h^{1/2}\Delta=\text{const} \qquad (8)$$

which, in the transition to the wave height H, gives a known Green's law

$$H=H_0(h/h_0)^{-1/4}(\Delta/\Delta_0)^{-1/2} \qquad (9)$$

where H_0, h_0, and Δ_0 are initial parameters of the corresponding quantities.

Mention should also be made of conservation laws for Eq. (7). It is known that there exists an infinite number of conservation laws for the Korteweg-de Vries equation, which is indicative of its full integrability (Whitham, 1974). A modified Korteweg-de Vries equation (7) possesses only two conservation laws:

$$h^{1/4}\Delta^{1/2}\int\eta(s,\ l)\,ds=\text{const} \qquad (10)$$

and

$$h^{1/2}\Delta\int\eta(s,\ l)\,ds=\text{const} \qquad (11)$$

which indicates that the energy and momentum are constant(integrals are taken with respect to period of wind waves). The first intergral is trivial: mean and unperturbed values of the water level coincide, while the second integral enables one to obtain analytical expressions for the variations in the wave height in the coastal zone, provided some suppositions on the wave shape are made. Thus, if the wave persists to be quasi-sinusoidal, its amplitude changes according to the Green's law; and if the wave is close to a periodic sequence of solitons, its amplitude is determined from a "nonlinear" analog of the Green's law(Ostrovsky and Pelinovsky, 1975):

$$H=H_0(h/h_0)^{-1}(\Delta/\Delta_0)^{-2/3} \qquad (12)$$

With a sufficiently slow(adiabatic) depth variation, which is true when the last term in Eq. (7) is small compared to the other terms, the wave is smoothly transformed as a cnoidal wave and its parameter variation was calculated by Ostrovsky and Pelinovsky (1970, 1975). Finally, the last analytical example emerges if the effect of dispersion is neglected (Ostrovsky and Pelinovsky, 1975; Pelinovsky, 1982). In this case, Eq. (7) is transformed into an equation for a simple wave:

$$\sqrt{gh}\frac{\partial\eta}{\partial l}+\frac{3\eta}{2h}\frac{\partial\eta}{\partial s}+\frac{\sqrt{gh}}{4h\Delta^2}\frac{d(h\Delta^2)}{dl}\eta=0 \quad (13)$$

Eq. (13) has an exact analytical solution

$$\eta(s,\ l)=H_0(h/h_0)^{-1/4}(\Delta/\Delta_0)^{-1/2}\zeta\Big\{t-\int\frac{dl}{(gh)^{1/2}}$$
$$-\eta(h/h_0)^{1/4}(\Delta/\Delta_0)^{1/2}\int(h/h_0)^{-7/4}(\Delta/\Delta_0)^{-1/2}dl\Big\} \quad (14)$$

where $\zeta(t)$ is the wave profile at the initial point. Using Eq. (14) one can determine the distance at which the wave will break. This case corresponds to $\partial\eta/\partial t$ vanishing to the infinity. In particular, if the wave propagates on a plane slope, it will break at the depth:

$$h_{\oplus}=h_0\{1+\alpha\lambda/2\pi H_0\}^{-4/3}, \quad (15)$$

This expression is valid if the wave breaks sufficiently far from the shore line.

The analytical examples presented here may be used(see the text below) for tests in working out numerical methods of integration of Eq. (7).

3. WAVE TRANSFORMATION AT CAUSTICS

When analysing wind wave transformation one must bear in mind the following significant circumstance. The formulas presented above are valid only when the rays do not intersect (Δ dose not turn to be zero). Such situations are typical in the coastal zone with simple geometry(straight and parallel depth contours) when the wave is moving in the region of decreasing depth. If the moving wave encounters regions of increasing depth, the rays will intersect due to refraction, thus forming caustics. The solution of linear problem of monochromatic wave field near smooth caustic is well known(see, for example, Mei (1989)). It shows that after passing caustics waves do not change in amplitude but do in phase on $\pi/2$. As a result, the form of periodical wave must change and it can be obtained by Fourier-superposition of monochromatic solutions. Corresponding formulae(Hibert transformation) is easily obtained in the frame of linear theory

$$\eta_{tr}(s,\ l)=\frac{1}{\pi}\int\eta_{in}(\tau,\ l)\cot\ \omega(s-\tau)d\tau \quad (16)$$

We have developed a method for the calculation of the nonlinear wave field in caustic regions (Pelinovsky, 1982; Engelbrecht *et al.*, 1988), which used linear solution in very narrow caustic zone having the size of order wave length. Weak nonlinear corrections to Eq. (16) are obtained in Engelbrecht *et al.*, (1988).

Eqs. (7) and (16) together with equations for rays make a complete system of equations of a nonlinear-ray theory of sea wave transformation in the coastal zone.

4. ACCOUNT OF DISSIPATION

To verify the practicability of the proposed model, consider the problem taking into account wind wave dissipation in the breaker zone. Wind waves are usually damped intensively in this region, and their energy consumed on transportation of sediment as well as on generation of storm surge and nearshore circulation. Theoretical models for the description of wind wave dissipation are rather diversified and are not quite reliable. With dissipation taken into account, Eq. (7), in a most general form, is modified as follows:

$$\sqrt{gh}\frac{\partial\eta}{\partial l}+\frac{3\eta}{2h}\frac{\partial\eta}{\partial s}+\frac{h}{6g}\frac{\partial^2\eta}{\partial s^2}+\frac{\sqrt{gh}}{4h\Delta^2}\frac{d(h\Delta^2)}{dl}\eta$$
$$=-\sqrt{gh}F_{fr} \quad (17)$$

We will describe a series of model theoretical expressions for wave dissipation. In the simplest models, the viscosity of water is taken into account by a dissipative force-friction:

$$F_{fr}=\frac{3\nu}{4h^2}\eta \quad (18)$$

where ν is molecular or turbulent viscosity of water. Since the flow under water crest is turbulent as a rule, the force of friction is approximated by the Chézy formula

$$F_{fr}=\mu g\eta|\eta|/2h\sqrt{gh}, \quad (19)$$

where μ is an empirical coefficient. Equations of this from have been widely used in the theory of stationary flows as applied to problems of river hydraulics. However, the coefficient μ is not constant for waves, which stipulates quite a hierarchy of dissipation models. Such models were used by Leontjev (1989) for the estimation of dissipation at breaking. Apparently, it is most correct to describe these process employing equations taking into account turbulent effects. Numerous examples of applications of the corresponding technique are given in, Voltzinger *et al.* (1989). This approach, however, implies use of three-dimensional equations and their numerical solution needs sufficiently powerful computers. For the purpose of application it is expedient to develop simplified models that allow for calculations on personal computers. In this case, basic equations must be of the form of Eq. (17).

Dissipation of the third type is related to wave scattering at a rough bottom (Dyatlov and Pelinovsky (1990)):

$$F_{fr} = -\delta \frac{\partial^2 \eta}{\partial s^2}, \qquad (20)$$

where $\delta = (h_{av}/h)k^2 l_0 (h^2/\sqrt{gh})$, with h_{av} being the average height of bottom inhomogeneities and l_0, the characteristic scale of the inhomogeneities. Analogous expressions are sometimes used for the description of undular bores.

The expressions for dissipative terms in Eqs. (18)-(20) do not increase the order of the differential equation (17) and consequently, do not affect cardinally the stability of developed numerical schemes.

5. NUMERICAL ALGORITHM

Our numerical modeling of wind wave transformation in the coastal zone is performed within a physical-mathematical model based on a modified Korteweg-de Vries equation of the form (17). It is convenient to omit the last term in the left-hand side of this equation and, instead, substitute

$$\zeta = \eta M, \ M = (h\Delta^2)^{1/4}/(h_0\Delta_0^2)^{1/4}, \qquad (21)$$

where h_o and Δ_o are the initial values of depth and ray beam cross-section.

It is easy to see that the quantity M corresponds to Green's law, if nonlinearity and dispersion are neglected. Therfore, the changes in the wave amlitude, ζ, will be readily interpreted in new variables, these changes may be related either to joint effect of nonlinearity and dispersion or to dissipation. We can now write an equation for the function ζ

$$\sqrt{gh}\frac{\partial \zeta}{\partial l} + \frac{3\zeta}{2hM}\frac{\partial \zeta}{\partial s} + \frac{h}{6g}\frac{\partial^3 \zeta}{\partial s^3} = -\sqrt{gh}F_{fr} \qquad (22)$$

Designating the coefficients as

$$\alpha = \frac{3}{2hQ\sqrt{gh}}, \ \beta \frac{h}{6g\sqrt{gh}} \qquad (23)$$

Eq. (22) takes a form of a standard Korteweg-de Vries equation with the right-hand side

$$\frac{\partial \zeta}{\partial l} + \alpha \frac{\partial \zeta}{\partial s} + \beta \frac{\partial^3 \zeta}{\partial s^3} = -F_{fr} \qquad (24)$$

The coefficients α and β are not constant here, but they depend on the depth of the basin and on the curvature radius of wave rays (i.e. on the cross-section of the ray beam). Eq. (24) is the object of our numerical modeling.

The Korteweg-de Vries equation with constant coefficients may be solved empolying finite-difference schemes that were surveyed by Berezin (1982). Those schemes, slightly modified, may also be used for the solution of Eq. (24) with variable coefficients. We will take a three-level explicit finite-difference scheme. Derivatives with respect to l and s will be replaced by the following differences:

$$\frac{\partial \zeta(s_j, l_j)}{\partial l} \to \frac{\zeta_j^{n+1} - \zeta_j^{n-1}}{2H} \qquad (25)$$

$$\frac{\partial \zeta(s_j, l_j)}{\partial s} \to \frac{\zeta_{j+1}^{n} - \zeta_{j-1}^{n}}{2T} \qquad (26)$$

$$\frac{\partial^3 \zeta(s_j, l_n)}{\partial s^3} \to \frac{\zeta_{j+2}^{n} - 2\zeta_{j+1}^{n} + 2\zeta_{j-1}^{n} - \zeta_{j-2}^{n}}{2T^3} \qquad (27)$$

In this representation s_j and l_n are the coordinates of numerical mesh points, ζ_j^n is the value of the function at the corresponding mesh points, while T and H are, respectively, the temporal and spatial steps of the numerical mesh. Eq. (24) may be represented in a finite-difference form

$$\frac{\zeta_j^{n+1}-\zeta_j^{n-1}}{2H}+\alpha_j\zeta_j^n\frac{\zeta_{j+1}^n-\zeta_{j-1}^n}{2T}$$
$$+\beta_j\frac{\zeta_{j+2}^n-2\zeta_{j+1}^n+2\zeta_{j-1}^n-\zeta_{j-2}^n}{2T^3}=-F_{frj} \qquad (28)$$

The error within which the difference equation (28) approximates the differential equation (24) can be determined in a differential approximation by expanding individual terms in Eq. (28) using Taylor series in the neighborhood of the point s_j, l_n and substituting these expressions into Eq. (28). Comparison with Eq. (24) shows that Eq. (28) approximates Eq. (24) at a rather smooth variation of α and β with the approximation of order T^2 and H^2 (Berezin, 1982). For Eq. (27) to reduce to Eq. (24) the stability condition must be met besides approximation. The stability criterion was obtained by Berezin (1982) who linearized the difference scheme and introduced fixed coefficients. Let us replace α and β in Eq. (28) as well as the factor ζ_j^n by average values α_0, β_0 and ζ_0. Berezin (1982) proved that the difference scheme for the solution of the Korteweg-de Vries equation is stable if the relation

$$\frac{H}{T}\left(\alpha_0\zeta_0+\frac{3\sqrt{3\beta_0}}{2T^2}\right)\leq 1 \qquad (29)$$

is met for the quantities T and H. It was pointed out (Berezin, 1982) that the relation, Eq. (29), is fairly exact, even minor deviations from this condition provoke the development of instability. It should be noted that the coefficients α and β change in the coastal zone due to variable depth and to the narrowing (or broadening) of the ray tube. Besides, the wave amplitude in Eq. (28) may also vary along the propagation path, therefore the relation, Eq. (28) must be fulfilled along the entire path of calculations. For a relatively small step T and $\zeta\sim 1$, the first term in the brackets in Eq. (29) is small and the condition is simplified to the form

$$H\leq 0.384T^3/\beta_0 \qquad (30)$$

This critierion may be used in most cases, except the neighborhood of caustics and the shore-line region. These regions must, naturally, be identified as early as at the stage of ray construction and need alternative approaches such as those described in section 3.

The initial conditions for Eq. (24) are specified for $l=l_0$ where l_0 is an arbitrary point on the ray, which determines its initial location. The function

$$\zeta(s,\ l_0)=\eta_0(t) \qquad (31)$$

that corresponds to the record of the wave at this poing must be known. Designate the record length through RL. In our numerical computations we assume the ζ function to be periodic with respect to s(this is always possible for the functions specified on a section) and solve a periodic problem in the interval RL. Since in a three-level numerical scheme the calculation of the function at the $n+1$ level needs knowledge of its values at the n and n-1 levels, the zero level of the mesh is filled by fixed initial conditions at the start of computations. The first level is determined by means of a simpler two-level scheme with the approximation order T and H^2.

Another peculiarity of the scheme is that it conserves to a great accuracy for $F_{fr}=0$ two integrals

$$I_1=\frac{1}{RL}\int_0^{RL}\zeta(s,\ l)ds \qquad (32)$$

$$I_2=\frac{1}{RL}\int_0^{RL}\zeta^2(s,\ l)ds \qquad (33)$$

Integral invariants in our computations are conserved to an accuracy of 10^{-3} %. In the presence of dissipation, the energy of wave motions is conserved no longer. It always decreases, which does not interfere with the increase of the wave amplitude in different portions with decreasing depth. As to the first integral, it may be conserved for some types of dissipation, for example, if friction is described by expressions of the form (20). Numerical experiments have confirmed these theoretical predictions.

The scheme presented above was verified on test examples with soliton propagation in a basin of constant depth as well as of variable depth when Eq. (21) allows for analytical solution(these problems are of particular interest for the problems of practical application such as the dynamics of tsunami waves (Pelinovsky, 1982). Besides, the results were compared with data of soliton hydromodeling as well as of numerical modeling of solitons within more complete models (some of them were described by Pelinovsky (1982)). Comparison was also made with data of real observations of tsunami on 26 May, 1983 in the Sea of Japan where soliton-like waves were formed. However, the latter is only a qualitative comparison because we lack data on the initial shape of the tsunami wave. Thus, we arrive at a conclusion that the part of physical-mathematical model that is related to the Korteweg-de Vries equation is quite efficient and reliable. The principal difficulties in the dissipative part of the model are caused by insufficient knowledge of the mechanisms responsible for dissipative processes. We have already mentioned that reliable models must contain parts for the solution of equations for turbulent energy. These parts, however, are not adapted to personal computers yet. Alternative approximations of dissipative terms contain many empirical constants and are not reliable enough. The prospects of such models may be evaluated only from comparison with reliable experimental data. Available model concepts of dissipative terms can readily be realized numerically because the dissipative terms do not exceed the order of the differential equation. In particular, if the dissipation is described by the expression (20), it can be approximated numerically as

$$\delta_i = \frac{\zeta_{j+1}^n - \zeta_j^{n+1} - \zeta_j^{n-1} + \zeta_{j-1}^n}{T^2}, \tag{34}$$

which is referred to as the DuFort-Frankel scheme (MacKraken and Dorn, 1975). It is known that in combination with Eq. (26) this scheme (neglecting nonlinearity and dispersion) is absolutely stable (MacKraken and Dorn, 1975). The presence of dissipation also affects, in principle, the stability of numerical scheme but if the dissipation is relatively small, its effect on the stability of the scheme is insignificant too, which was proved rigorously. Consequently, the step of the numerical mesh was chosen according to the criterion, Eq. (29). In other cases, when the effect of dissipation was pronounced enough, the step in l was diminished unless computations with different steps coincided.

6. NUMERICAL MODELING

Consider results of a series of numerical experiments on the evolution of regular wind waves. A periodic problem is readily formulated in this case. The computations were performed for comparable nonlinear and dispersion effects. The Ursell parameter

$$U_r = \frac{9\alpha_0\lambda_0^2}{h^3} \tag{35}$$

was a criterion in choosing the necessary situations. (Here α_0 is the initial amplitude of the wave of length λ_0.) If the Ursell parameter is small, dispersion effects dominate and the wave retains almost a sinusoidal shape. While for a large Ursell parameter, nonlinear effects transform the wave profile substantially, thus competing with dispersion effects. It is clear from Eq. (35) that the region of moderate values of U_r that are interesting to us corresponds to small depths of about 10 m, given the wave amplitude of 1 m and its period 6 s, which is typical of wind waves, These parameters are used in our calculations that will be presented below.

6.1 Uniform Bottom

Let the wave period be 6 s, amplitude 1 m and the basin depth 1.45 m. Then, $U_r=40$ and pronounced nonlinear effects may be expected. Fig. 1a depicts the variation of the wave record at the initial stage when the wave has propagated 3.6 m and 7.3 m. One can see that the wave profile changes significantly: solitons emerge in the wave crest. The number of solitons is determined roughly by the Ursell number-to-12 ratio (12 being the characteristic of a single soliton) and in our case is equal to 3 like in Fig. 1a. The wave amplitude increases sharply (more than two fold). Later the solitons move with a velocity that is higher than $\sqrt{gh}$ and depends on the soliton amplitude as was predicted theoretically (Whitham, 1974). This motion is well seen in Fig. 1b which displays wave records past 11 m, 15 m, and 18 m. At still greater distances, large-amplitude solitons overtake smaller-amplitude solitons. The wave again takes a nearly sinusoidal shape at this moment. This process is illustrated in Fig. 1c for the wave that has covered the distance of 57 m. The process of recovery of initial state

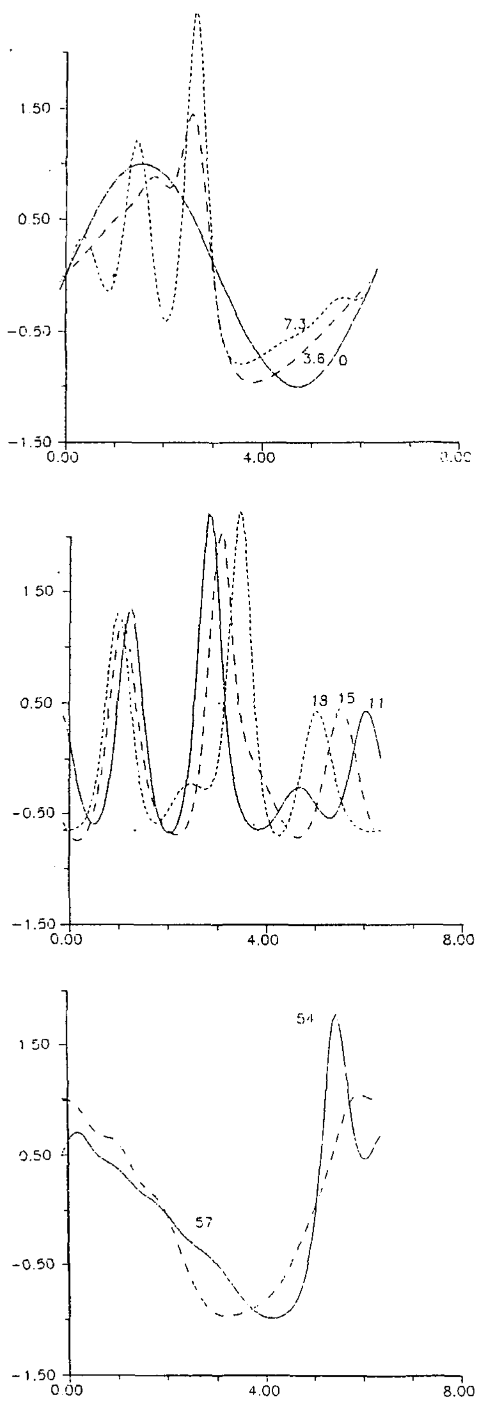

Fig. 1(a, b, c). Nonlinear transformation of a sinusoidal wave in basin of const depth ($h=1.45$ m ($T=$s, $\eta_0=1$ m). Here and in other figures the numbers at the curves indicate the distance (in meters) passed by the wave.

(recurrence) is well known for the Korteweg-de Vries equation and is related to its full integrability (Whitham, 1974).

The given example is, on the one hand, a test which demonstrates the effciency of the numerical mesh for recurrence. On the other hand, it illustrates pictorially the importance of taking into account nonlinear effects in shallow waters that are one of the sources of wave amplitude variations and the changes of the wave shape (and the wave spectrum consequently) in the coastal zone.

6.2 Decreasing Depths

Consider the transformation of a monochromatic wave in the region of decreasing depth. Take as an example the following law of depth variation:

$$h(x)=9-2.5[th\frac{x-150}{50}+1] \qquad (36)$$

which corresponds to smooth transitions of the wave with a period 6 *s* and amplitude 1 m from the 9 m to the 4 m depths, the size of the transition zone being about 100 m. Wave records in "deep" water at the distances of 30 m and 60 m from the initial location (the distances from the center of depth difference) are shown in Fig. 2a. Because the values of depth are large and the propagated path is short, the shape of the wave changes only slightly. As the wave enters the transition region its shape alters substantially. This process is illustrated in Fig. 2b where wave profiles are shown at the distances of 180 m and 210 m from the source (30 m and 60 m past the center of the transition region). The wave amplitude ζ increases by 1.4 times. The Green factor (22) must be taken into account in calculations for the real wave height η, which gives a nearly 50% increase of true wave amplitude. The depth of the trough changes too: it drops by 30% for the ζ function and by about 20% for the true variable η. This example, in addition, illustrates the advantages of eliminating the Green factor for the interpretation of the observed wave height variation that now depends only on nonlinearity and dispersion. Nonlinearity and dispersion are significant in shallow water 4 m deep while the shape of the wave and its amplitude change strongly in the course of

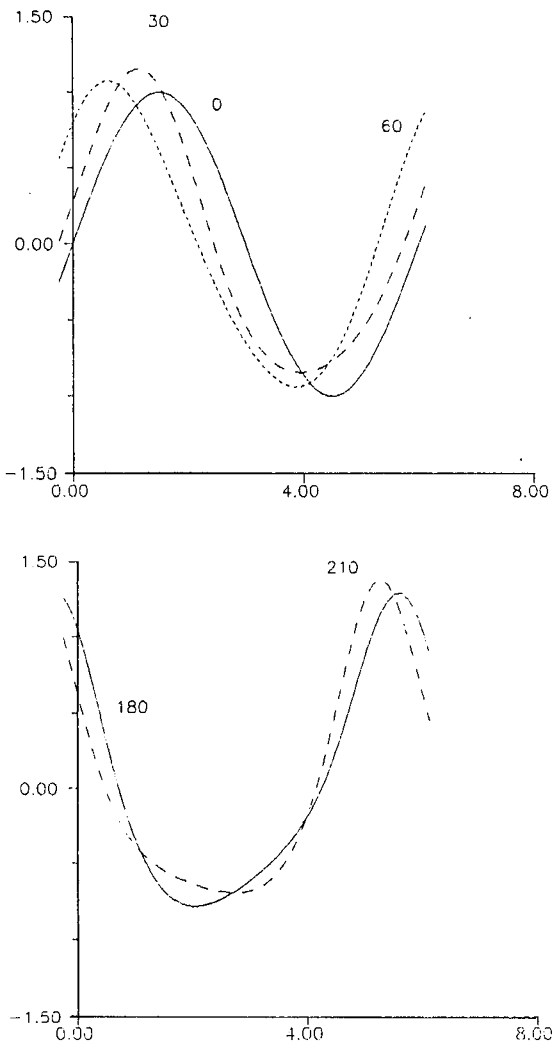

Fig. 2(a, b). Wave transition from the depth of 9 m to the depth of 4 m (T=6s, η_0=1 m).

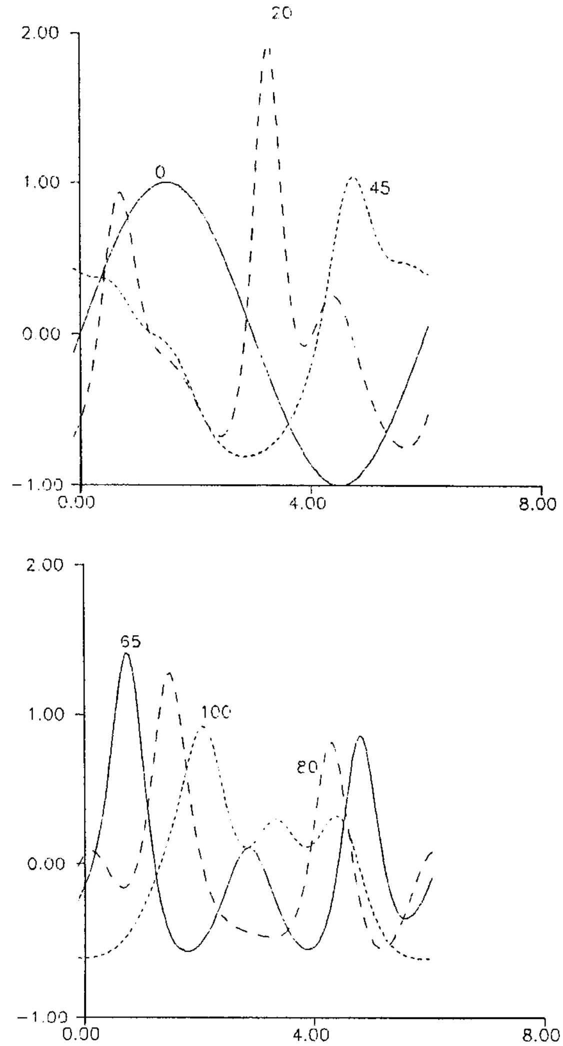

Fig. 3(a, b). The effect of viscosity on the propagation of a sinusoidal were at the depth of 1.45 m (T=6 s, η_0=1 m, δ=0.1 m^2/s).

propagation.

6.3 Account of Viscous Dissipation

We will now demonstrate the effect of viscous dissipation caused by wave scattering at a rough bottom. In this case, the dissipative factor is described by Eq. (20). The effective viscosity factor is chosen to be equal to δ=0.1m^2/s, which corresponds to turbulent viscosity. The wave period (6s) and the basin depth (1.45 m) are taken the same as in the case of uniform bottom to provide dominating nonlinearity. The initial stage of wave evolution at distances up to 45 m is shown in Fig. 3a. We can see that the wave shape changes substantially and solitons with the amplitudes amounting to double initial values are generated in the wave. At larger distances (Fig. 3b), the wave persists to be nonsinusoidal and its amplitude damps, although rather slowly: the wave amplitude at a distance of 100 m is only slightly smaller than its initial value. Nevertheless, direct calculation of the integral (33) shows that the wave energy decreases by about 2.5 times. Within a linear theory, the wave amplitude would decrease only by 1.5 times at the same distance. Thus, our example demonstrates that nonlinearity effectively competes with dispersion if we are concerned only with amplitude variation.

The example presented above is a model one to a great extent. We have already mentioned that other types of dissipation may also be taken into account, provided that they do not exceed the order of the KdV equation and contain the wave field rather than its integrals, for instance, energy like in most empirical models. We believe, however, that the dissipative part of the model is not yet reliable enough.

7. CONCLUSION

Let us now sum up results obtained in this paper.

1. A physical-mathematial model for wave transformation in the coastal zone based on a modified Korteweg-de Vries equation has been described. The rays in this model are determined from a linear theory of long waves and the wave amplitude, from an evolution Korteweg-de Vries equation. The model is supplemented with a part that allows for the calculation of wave transformation in the caustic zone where the rays intersect.

2. A numerical scheme for the integration of the Korteweg-de Vries equation with variable coefficients has been developed.

3. Wave transformation has been calculated. The calculations show the nonlinear and dissipative effects at even and rough bottom. The model has been tested on known analytical solutions of soliton evolution and recurrence of periodic perturbations.

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