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Peak Distribution of Nonlinear Random Waves of Finite Bandwidth

有限한 Spectral Bandwidth를 갖는 Nonlinear Random Waves의 推計學的 性質

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Abstract □ The theoretical treatment of statistical properties and distribution relevant to nonlinear random wave field of moderate bandwidth such as peak and trough of wave elevation is an overdue task hampered by the complicated form of nonlinear random waves. In this study, the extreme distribution of nonlinear random waves is derived based on the simplified version of Longuet-Higgins' wave model. It is shown that the band width of wave spectrum has a significant influence on these extreme distribution and the significant wave height is getting larger in an increasing manner as the nonlinearity is getting profound.

要 旨: 風波가 그 生成領域을 벗어나 전이해 갈 경우 스펙트럼의 폭은 side band instability 등으로 인해 증가하게 되는데 이 경우 非線形不規則波의 推計學的 性質에 대해서는 거의 糾明된 바 없으나 간략화된 Longuet-Higgins의 파랑모형을 토대로 비선형 불규칙파의 파봉에 대한 確率密度函數를 유도하였으며 모의 결과 스펙트럼 폭이 peak의 分布에 지대한 영향을 미치며 非線形의 程度가 심화될 경우 有義波高는 增加하는 것으로 밝혀졌다.

1. INTRODUCTION

Past descriptions of statistical properties of wind waves are mostly based on the assumption that waves are linear and Gaussian. Under mild wind conditions, linear wave theory has been shown to be a robust model in many cases. However, waves generated by high winds are steep and nonlinear, and wave-wave interaction may become important. Another assumption which has on occasions been made is that waves are narrow banded. Again, in mild seas, without considering contamination due to swell, this assumption has been shown to be reasonably accurate. In high seas, however, this assumption may not be satisfactory. Ocean facilities are designed against severe events in which case waves are steep, and their frequency spectrum is not necessarily narrow. To improve wave model for these circumstances, it is first of all necessary to relax the requirement of narrow band assumption. Secondly, a nonlinear wave model must be established. For waves of single frequency, nonlinear waves can be described by the Stokes waves based on the assumption that the slope of the wave is small and perturbation technique can be used. For waves consisting of many components, Longuet-Higgins (1963) has used the perturbation technique to obtain an expression for wave elevation to the second order of approximation. Based on the expression, he used the Gram-Charlier series to describe the probability density function of wave elevation. It was shown that the probability density function of wave elevation deviates from the Gaussian probability density function as can be expected. This technique was later extended by Jackson (1979) to give the joint probability density function of wave elevation and

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wave slope. There are, however, certain shortcomings; the use of the Gram-Charlier series gives negative probability density at high values of elevation and slope, and the use of wave elevation given by Longuet-Higgins (1963) to obtain properties of the sea surface can be mathematically inconvenient. Tayfun (1980) obtained an expression for nonlinear wave elevation using the Stokes perturbation technique for waves of narrow bandwidth. Later, Huang *et al.* (1983) extended the method of Tayfun for narrow banded waves to the third order and obtained expressions of the probability density function of wave elevation, slope, as well as the joint probability density function of wave elevation and slope (Huang *et al.*, 1983). The search for a way simpler than that of Longuet-Higgines (1963) to describe nonlinear waves of finite bandwidth was recently carried out by Tung *et al.* (1989). Based on the studies of Tayfun (1980, 1983), Tung *et al.* (1989) proposed a simple but accurate expression for second order nonlinear wave elevation for waves of moderate bandwidth.

With this wave model available, it is the intent of this study to obtain, for second order nonlinear waves, statistical properties such as the probability density function of peaks, quantities that are of great importance in the design of ocean structures. In this study, our attention is centered on deep water waves only.

2. REVIEW OF EXTREME DISTRIBUTION THEORY

It is known that for stationary random process $\zeta(t)$ of arbitrary bandwidth, the probability density function of peaks is

$$f(\zeta_o)=F_p(\zeta_o)/N_p \tag{1}$$

where

$$F_p(\zeta_o)=\int_{-\infty}^{0}|\ddot{\zeta}|f_{\zeta\dot{\zeta}\ddot{\zeta}}(\zeta_o,\ 0,\ \ddot{\zeta})d\ddot{\zeta} \tag{2}$$

$$N_p=\int_{-\infty}^{\infty}\int_{-\infty}^{0}|\ddot{\zeta}|f_{\zeta\dot{\zeta}\ddot{\zeta}}(\zeta_o,\ 0,\ \ddot{\zeta})d\zeta_o d\ddot{\zeta} \tag{3}$$

and the probability density function of troughs is given by

$$f(\zeta_o)=F_t(\zeta_o)/N_t \tag{4}$$

where

$$F_t(\zeta_o)=\int_{0}^{\infty}|\ddot{\zeta}|f_{\zeta\dot{\zeta}\ddot{\zeta}}(\zeta_o,\ 0,\ \ddot{\zeta})d\ddot{\zeta} \tag{5}$$

$$N_t=\int_{-\infty}^{\infty}\int_{0}^{\infty}|\ddot{\zeta}|f_{\zeta\dot{\zeta}\ddot{\zeta}}(\zeta_o,\ 0,\ \ddot{\zeta})d\zeta_o d\ddot{\zeta} \tag{6}$$

In (2), (3), (4) and (5), $f_{\zeta\dot{\zeta}\ddot{\zeta}}(\cdot,\cdot,\cdot)$ is the joint probability density function of ζ, $\dot{\zeta}$ and $\ddot{\zeta}$, and $F_p(\zeta_o)$ represents the mean number of occurrences per unit time for which a peak of ζ takes the value between ζ_o and $\zeta_o+d\zeta_o$ and N_p is the mean number of occurrences of a peak of ζ per unit time regardless of magnitude. To apply (1) and (4) to nonlinear random waves, it is necessary to have the joint distribution of ζ, $\dot{\zeta}$ and $\ddot{\zeta}$ which in turn requires a nonlinear wave model of finite bandwidth.

3. FINITE BAND APPROXIMATION

Longuet-Higgines (1963) gave a second order deep water nonlinear Stokes wave model for random waves of arbitrary bandwidth. The expression for wave elevation is given by

$$\zeta=\sum_{i=1}^{\infty}a_i\cos\chi_i+\frac{1}{2g}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_ia_j\omega_i^2\cos(\chi_i+\chi_j)-\frac{1}{2g}\sum_{i=1}^{\infty}\sum_{j>1}^{\infty}a_ia_j(\omega_j^2-\omega_i^2)\cos(\chi_j-\chi_i) \tag{7}$$

in which $\chi_i=k_i\chi-\omega_it+\varepsilon_i$, k_i is the wave number, $\omega_i=(gk_i)^{1/2}$ is wave frequency, ε_i is random phase uniformly distributed over the interval $(0,\ 2\pi)$ and a_i is the amplitude of the component wave. It is obvious that straightforward use of (7) to obtain the joint probability density function of ζ, $\dot{\zeta}$ and $\ddot{\zeta}$ would be rather cumbersome. Tayfun (1986) examined (7) closely and showed that the third term on the right hand side of (7) contains little energy compared with the first two terms. Based on this observation, Tung *et al.* (1989) elected to neglect the third term altogether and obtained a nonlinear wave model which is more convenient to use. Upon introducing the following random processes

$$\eta_1=\frac{1}{(M_0)^{1/2}}\sum_{i=1}^{\infty}a_i\cos\chi_i \tag{8}$$

$$\eta_2 = \frac{1}{(M_2)^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i \sin \chi_i \quad (9)$$

$$\eta_3 = -\frac{1}{(M_4)^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i^2 \cos \chi_i \quad (10)$$

$$\eta_4 = -\frac{1}{(M_4)^{1/2}} \sum_{i=1}^{\infty} a_i \omega_i^2 \sin \chi_i \quad (11)$$

$$\eta_5 = \frac{1}{(M_0)^{1/2}} \sum_{i=1}^{\infty} a_i \sin \chi_i \quad (12)$$

it was shown that the nondimensional nonlinear wave elevation, ζ_1, can be written as

$$\zeta_1 = \zeta/(m_0)^{1/2} \doteq \left(\frac{M_0}{m_0}\right)^{1/2} \left(\eta_1 - \frac{1}{2}\varepsilon\eta_1\eta_3 + \frac{1}{2}\varepsilon\eta_4\eta_5\right) \quad (13)$$

where M_i and m_i are the ith spectral moments of the linear and nonlinear wave elevation, respectively and $\varepsilon = (M_4)^{1/2}/g$. For a monochromatic wave of amplitude a and frequency ω, $M_4 = a^2\omega^4/2$ so that $\varepsilon = ak/2$ is a small quantity. For the problem under consideration, ε will be used as a perturbation parameter.

4. JOINT DISTRIBUTION OF A NONLINEAR WAVE ELEVATION, ITS FIRST AND SECOND DERIVATIVE

Our task is to obtain the joint distribution of nonlinear wave elevation, its first and second derivatives to be used in (1) and (4). To this end, we carry out the differentiation of nonlinear wave elevation with respect to time twice. The resulting expressions of the nondimensionalized first and second derivatives of nonlinear wave elevation involve many random variables. Although the joint distribution of ζ, $\dot{\zeta}$ and $\ddot{\zeta}$, $f_{\zeta\dot{\zeta}\ddot{\zeta}}(\cdot)$, may be obtained, the task is tedious. To facilitate subsequent computation of $f_{\zeta\dot{\zeta}\ddot{\zeta}}(\cdot)$, we further introduce some assumptions based on the analysis given by Tayfun (1986). We first note that η_1 in (8) and η_5 in (12) may be written as

$$\eta_1 = \frac{1}{M_o^{1/2}} C \cos(k_c x - \omega_c t - \phi) \quad (14)$$

$$\eta_5 = \frac{1}{M_o^{1/2}} C \sin(k_c x - \omega_c t - \phi) \quad (15)$$

where k_c and ω_c are the spectral mean wave number and frequency, respectively and $C(x,t)$ and $\phi(x,t)$ are the amplitude and phase processes. Furthermore, we note that

$$\eta_2 = \dot{\eta}_1 \left(\frac{M_0}{M_2}\right)^{1/2} \quad (16)$$

$$\eta_3 = \ddot{\eta}_1 \left(\frac{M_0}{M_4}\right)^{1/2} \quad (17)$$

$$\eta_4 = \ddot{\eta}_5 \left(\frac{M_0}{M_4}\right)^{1/2} \quad (18)$$

It was shown by Tayfun (1986) that if C and ϕ are $O(1)$, then $\dot{C}$ and $\dot{\phi}$ are $O(\nu)$ and all higher order derivatives of C and ϕ are of corresponding higher order smallness of ν where

$$\nu = ((M_0 M_2/M_1^2) - 1)^{1/2} < 1 \quad (19)$$

is a measure of the bandwidth of the frquency spectrum which, for all practical purposes, is a small quantity. Based on this assumption, nondimensional wave elevation, ζ_1, can be rewritten, to the order of ν,

$$\begin{aligned} \zeta_1 &= \left(\frac{M_0}{m_0}\right)^{1/2} \left(\eta_1 - \frac{1}{2}\varepsilon\eta_1\eta_3 + \frac{1}{2}\eta_4\eta_5\right) \\ &\doteq \left(\frac{M_0}{m_0}\right)^{1/2} \Big\{C\cos\chi - \frac{1}{2}\varepsilon C \cdot C(\omega_c + \dot{\phi})^2 \\ &\quad (-\cos\chi\cos\chi + \sin\chi\sin\chi)\Big\} \\ &\doteq \left(\frac{M_0}{m_0}\right)^{1/2} \Big\{C\cos\chi + \frac{1}{2}\varepsilon C(\omega_c + \dot{\phi})^2 \cos 2\chi\Big\} \end{aligned} \quad (20)$$

Then it follows that, to the order of ν,

$$\begin{aligned} \zeta_2 &= \dot{\zeta}_1 \left(\frac{m_0}{m_2}\right)^{1/2} \\ &\doteq \left(\frac{M_2}{m_2}\right)^{1/2} \Big\{\dot{C}\cos\chi + C(\omega_c + \dot{\phi})\sin\chi \\ &\quad + \varepsilon C(\omega_c + \dot{\phi}) \cdot C(\omega_c + \dot{\phi})^2 \sin 2\chi\Big\} \\ &\doteq \left(\frac{M_2}{m_2}\right)^{1/2} \Big\{\dot{C}\cos\chi + C(\omega_c + \dot{\phi})^2 \sin\chi \\ &\quad + 2\varepsilon C(\omega_c + \dot{\phi})\sin\chi \cdot C(\omega_c + \dot{\phi})^2 \cos\chi\Big\} \end{aligned} \quad (21)$$

$$\begin{aligned} \zeta_3 &= \ddot{\zeta}_1 \left(\frac{m_0}{m_4}\right)^{1/2} \\ &\doteq \left(\frac{M_4}{m_4}\right)^{1/2} \Big\{-C(\omega_c + \dot{\phi})^2 \cos\chi \\ &\quad - 2\varepsilon C(\omega_c + \dot{\phi})^2 \cdot C(\omega_c + \dot{\phi})^2 \cos 2\chi\Big\} \end{aligned} \quad (22)$$

$$\doteq \left(\frac{M_4}{m_4}\right)^{1/2}\Big\{-C(\omega_c+\dot{\phi})^2\cos\chi$$
$$-2\varepsilon C^2(\omega_c+\dot{\phi})^4\cos^2\chi+2\varepsilon C^2(\omega_c+\dot{\phi})^4\sin^2\chi\Big\}$$

In terms of η_i, ζ_2 and ζ_3 are

$$\zeta_2 \doteq \left(\frac{M_2}{m_2}\right)^{1/2}(\eta_2-2\varepsilon\eta_2\eta_3) \tag{23}$$

$$\zeta_3 \doteq \left(\frac{M_4}{m_4}\right)^{1/2}(\eta_3-2\varepsilon\eta_3^2+2\varepsilon\eta_4^2) \tag{24}$$

In (13), (23) and (24), to the first order of ε, $m_0=M_0$, $m_2=M_2$ and $m_4=M_4$. The random variables, ζ_1, ζ_2 and ζ_3 are seen to be functions of η_1, η_2, η_3, η_4 and η_5 which are random variables having zero mean and unit standard deviation. Furthermore, the pairs (η_1, η_3) and (η_2, η_4, η_5) are statistically independent, each of which is jointly Gaussian. Therefore the joint distribution of η_1, η_2, η_3, η_4 and η_5 is given by

$$f_{\eta_1\eta_2\eta_3\eta_4\eta_5}(\cdot,\cdot,\cdot,\cdot,\cdot)=f_{\eta_1\eta_3}(\cdot,\cdot)f_{\eta_2\eta_4\eta_5}(\cdot,\cdot,\cdot) \tag{25}$$

where

$$f_{\eta_1\eta_3}(\cdot,\cdot)=\frac{1}{2\pi\sqrt{1-\rho_1^2}}\exp\Big[-\frac{1}{2(1-\rho_1^2)}(\eta_1^2+\eta_3^2-2\rho_1\eta_1\eta_3)\Big] \tag{26}$$

$$f_{\eta_2\eta_4\eta_5}(\cdot,\cdot,\cdot)=\frac{1}{(2\pi)^{3/2}|S|^{1/2}}\exp\Big[-\frac{1}{2|S|}\sum_{j=1}^{3}\sum_{k=1}^{3}|S|_{jk}\eta_j\eta_k\Big] \tag{27}$$

where the correlation coefficient ρ_1 is $E[\eta_1\eta_3]$.

In (27), the matrix S of covariances of η_2, η_4 and η_5 is

$$S=\begin{vmatrix} E[\eta_2^2] & E[\eta_2\eta_4] & E[\eta_2\eta_5] \\ E[\eta_4\eta_2] & E[\eta_4^2] & E[\eta_4\eta_5] \\ E[\eta_5\eta_2] & E[\eta_5\eta_4] & E[\eta_5^2] \end{vmatrix} \tag{28}$$

and $|S|_{jk}$ is the cofactor of the element in the jth row and kth column of S. After denoting the correlation coefficients $E[\eta_4\eta_2]=\rho_2$ and $E[\eta_2\eta_5]=E[\eta_5\eta_2]=\rho_3$, S is

$$S=\begin{vmatrix} 1 & \rho_2 & \rho_3 \\ \rho_2 & 1 & \rho_1 \\ \rho_3 & \rho_1 & 1 \end{vmatrix} \tag{29}$$

By introducing the auxiliary random variables

$$\zeta_4=\eta_4 \tag{30}$$

$$\zeta_5=\eta_5 \tag{31}$$

the joint distribution of ζ_1, ζ_2, ζ_3, ζ_4 and ζ_5, $f_{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}(\cdot)$, can be obtained by the standard method of transformation of random variables [Papoulis, 1965].

This is,

$$f_{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}(\cdot)=f_{\eta_1\eta_3}(\eta_1,\eta_3)f_{\eta_2\eta_4\eta_5}(\eta_2,\eta_4,\eta_5)\left|J\left(\frac{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}{\eta_1\eta_2\eta_3\eta_4\eta_5}\right)\right|^{-1} \tag{32}$$

where J is the Jacobian of the variable transformation. From (13), (23), (24), (30) and (31), and following the perturbation technique used by Huang *et al.* (1983), it may be shown that, to the order of ε,

$$|J|=1-\frac{13}{2}\varepsilon\eta_3 \tag{33}$$

Here, η_1, η_2, η_3, η_4 and η_5 on the right hand side of (32) are to be replaced by the inverse of the functions in (13), (23), (24), (30) and (31). These inverses may also be obtained approximately by the perturbation method. That is, to the order of ε,

$$\eta_1 \doteq \zeta_1+\frac{1}{2}\varepsilon\zeta_1\zeta_3-\frac{1}{2}\varepsilon\zeta_4\zeta_5 \tag{34}$$

$$\eta_2 \doteq \zeta_2+2\varepsilon\zeta_2\zeta_3 \tag{35}$$

$$\eta_3 \doteq \zeta_3+2\varepsilon\zeta_3^2-2\varepsilon\zeta_4^2 \tag{36}$$

$$\eta_4=\zeta_4 \tag{37}$$

$$\eta_5=\zeta_5 \tag{38}$$

Substituting (34), (35), (36), (37) and (38) into (32) and performing the integration with respect to ζ_4 and ζ_5,

$$f_{\zeta_1\zeta_2\zeta_3}(\cdot,\cdot,\cdot)=\iint f_{\zeta_1\zeta_2\zeta_3\zeta_4\zeta_5}(\cdot,\cdot,\cdot,\cdot,\cdot)d\zeta_4 d\zeta_5 \tag{39}$$

the joint distribution of a nondimensional wave elevation, its nondimensional first and second derivatives can be obtained. The integration in (39) can be easily carried out although the task is lengthy and tedious. The result is

$$f_{\zeta_1\zeta_2\zeta_3}(\cdot,\cdot,\cdot)=\frac{Q}{(2\pi)^{3/2}\sqrt{1-\rho_1^2}}\exp\left\{-\frac{1}{2(1-\rho_1^2)}(\zeta_1^2+\zeta_3^2-2\rho_1\zeta_1\zeta_3)\right\}\cdot\exp\left(-\frac{1}{2}\zeta_2^2\right) \quad (40)$$

where

$$Q=Q_1+Q_2+Q_3 \quad (41)$$

$$Q_1=1+\frac{\varepsilon\rho_2}{2(1-\rho_1^2)}(\rho_3-4\rho_1\rho_2)\zeta_1\zeta_2^2-\frac{\varepsilon}{2(1-\rho_1^2)}\zeta_1^2\zeta_3-\frac{\varepsilon}{2(1-\rho_1^2)}(3\rho_1+\rho_2\rho_3-4\rho_1\rho_2^2)\zeta_1 \quad (42)$$

$$Q_2=\left\{\frac{\varepsilon\rho_2}{2(1-\rho_1^2)}(4\rho_2-\rho_1\rho_3)-2\varepsilon\right\}\cdot\zeta_2^2\zeta_3+\frac{\varepsilon}{2(1-\rho_1^2)}(17-14\rho_1^2+\rho_1\rho_2\rho_3-4\rho_2^2)\zeta_3 \quad (43)$$

$$Q_3=\frac{\varepsilon}{2(1-\rho_1^2)}5\rho_1\zeta_1\zeta_3^2-\frac{\varepsilon}{2(1-\rho_1^2)}4\zeta_3^3 \quad (44)$$

5. PEAK DISTRIBUTION

Substituting (40) into (1), we can obtain the probability density function of the peak of nonlinear wave elevation, $f_{\zeta 0}(\zeta_0)$.

That is, $f_{\zeta 0}(\zeta_0)$ is given by

$$f_{\zeta 0}(\zeta_0)=F_p(\zeta_0)/N_p \quad (45)$$

$$F_p(\zeta_0)=\int_{-\infty}^{0}|\zeta_3|f_{\zeta_1\zeta_2\zeta_3}(\zeta_0,\ 0,\ \zeta_3)d\zeta_3 \quad (46)$$

$$=\tilde{M}\int_{-\infty}^{-\rho_1\zeta_0/\sqrt{1-\rho_1^2}}\exp\left(-\frac{1}{2}z^2\right)dz+\tilde{N} \quad (47)$$

where

$$\tilde{M}=\frac{1}{(2\pi)^{3/2}}\left\{-\frac{\varepsilon}{2}(5-2\rho_1^2+\rho_1\rho_2\rho_3-4\rho_2^2)-\rho_1\zeta_0+\frac{\varepsilon}{2}\left(1-5\rho_1^2+\rho_1\rho_2\rho_3\right)\zeta_0^2+\frac{\varepsilon}{2}\rho_1^2\zeta_0^4\right\}\cdot\exp\left(-\frac{1}{2}\zeta_0^2\right) \quad (48)$$

$$\tilde{N}=\frac{1}{(2\pi)^{3/2}}\sqrt{1-\rho_1^2}\left\{1+\frac{\varepsilon}{2}(4\rho_1-\rho_2\rho_3)\zeta_0-\frac{\varepsilon}{2}\rho_1\zeta_0^3\right\}\exp\left(-\frac{1}{2}\frac{\zeta_0^2}{1-\rho_1^2}\right) \quad (49)$$

$$N_p=\frac{1}{2\pi}\left\{1-\varepsilon(1-\rho_2^2)\sqrt{2\pi}\right\} \quad (50)$$

To quantify the above results, we must specify the wave spectrum from which the quantities ρ_1, ρ_2, ρ_3 and ε may be calculated. In this study, we shall use the Wallops spectrum (Huang *et al.*, 1981) which takes the form

$$\Phi(\omega)=\frac{\alpha g^2}{\omega^m\omega_0^{5-m}}\exp\left[-\frac{m}{4}\left(\frac{\omega_0}{\omega}\right)^4\right] \quad (51)$$

where

$$m=\left|\frac{\log(2\pi^2\xi^2)}{\log 2}\right| \quad (52)$$

is the absolute value of the slope of the spectrum (on the log-log scale) in the high frequency range and

$$\xi=M_0^{1/2}/L_0=\sigma k/(2\pi)=\varepsilon/(2\pi) \quad (53)$$

is the significant slope, L_0 being the wave length whose frequency ω_0 corresponds to the peak of the single peak Wallops spectrum. In (51), the coefficient α is given by

$$\alpha=\frac{(2\pi\xi)^2 m^{(m-1)/4}}{4^{(m-5)/4}}\frac{1}{\Gamma((m-1)/4)} \quad (54)$$

where $\Gamma(\cdot)$ is the gamma function (Abramowitz and Stegun, 1968).

From (51), it may be shown that

$$\rho_1=-\frac{\Gamma[(m-3)/4]}{\Gamma[(m-1)/4]\Gamma[(m-5)/4]} \quad (55)$$

$$\rho_2=-\frac{\Gamma[(m-4)/4]}{\Gamma[(m-3)/4]\Gamma[(m-5)/4]} \quad (56)$$

$$\rho_3=-\frac{\Gamma[(m-2)/4]}{\Gamma[(m-1)/4]\Gamma[(m-3)/4]} \quad (57)$$

and

$$\varepsilon=2\pi\xi\left[\frac{m\Gamma[(m-5)/4]}{4\Gamma[(m-1)/4]}\right]^{1/2} \quad (58)$$

so that ε, ρ_1, ρ_2 and ρ_3 are solely dependent on the value of ξ which was shown (Huang *et al.*, 1980) to rarely exceed 0.02 in the ocean.

As $\varepsilon=0$, the peak distribution in (45) is reduced to

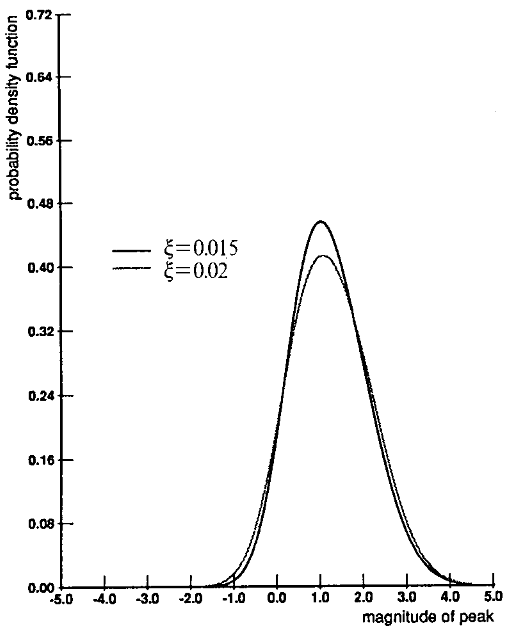

Fig. 1. Peak distribution.

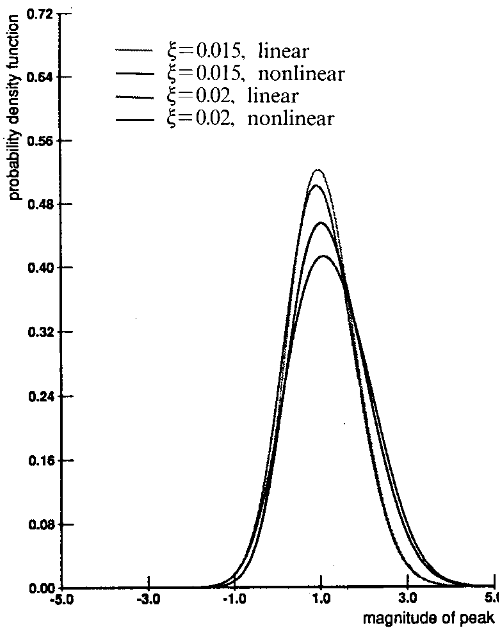

Fig. 2. Peak distribution.

$$f_{\zeta_0}(\zeta_0) = \frac{1}{\sqrt{2\pi}}\left\{-\rho_1\zeta_0\exp\left(-\frac{1}{2}\zeta_0^2\right)\cdot\int_{-\infty}^{-\rho_1\zeta_0/\sqrt{1-\rho_1^2}} \exp\left(-\frac{1}{2}z^2\right)dz+\sqrt{1-\rho_1^2}\cdot\exp\left(-\frac{1}{2}\frac{\zeta_0^2}{1-\rho_1^2}\right)\right\} \tag{59}$$

which was exactly the same as the probability density function of the peak of Gaussian process of finite bandwidth derived by Cartwright and Longuet-Higgins (1956). For a narrow-band process ($\rho_1 = -1$), (45) reduces to Rayleigh distribution. In Figs. 1 and 2, the probability density function in (45) is plotted for $\zeta=0.015$ and 0.02 and the peak distribution of linear waves is also included for comparison. It is noted that for random process of moderate bandwidth, the peaks can be negative as well as positive and the peak distribution associated with nonlinear waves differs from the linear counterpart. The general character of this difference is in the form of a spreading of the density mass toward the higher and lower crests and a negative skewness of peak distribution of nonlinear waves, whereas a positive skewness is detected in the peak distribution of a linear wave. In Figs. 3, the peak distributions of nonlinear waves of narrow bandwidth and moderate are plotted together for $\xi=0.015$.

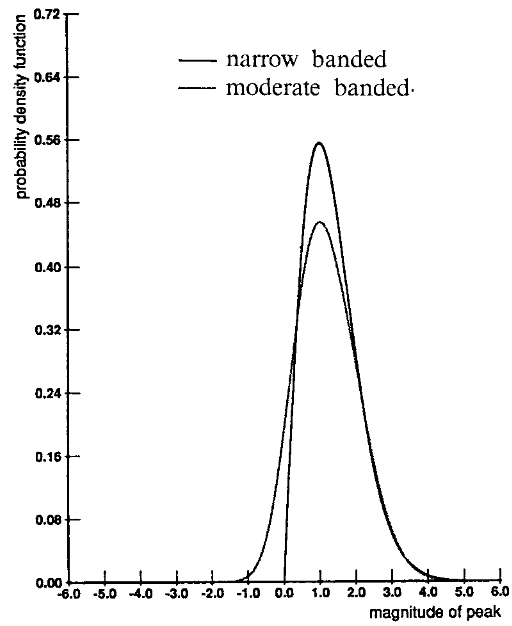

Fig. 3. Peak distribution (for $\xi=0.015$).

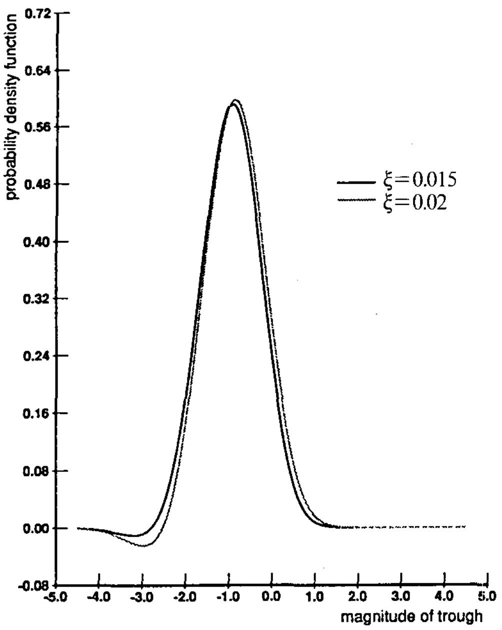

Fig. 4. Trough distribution.

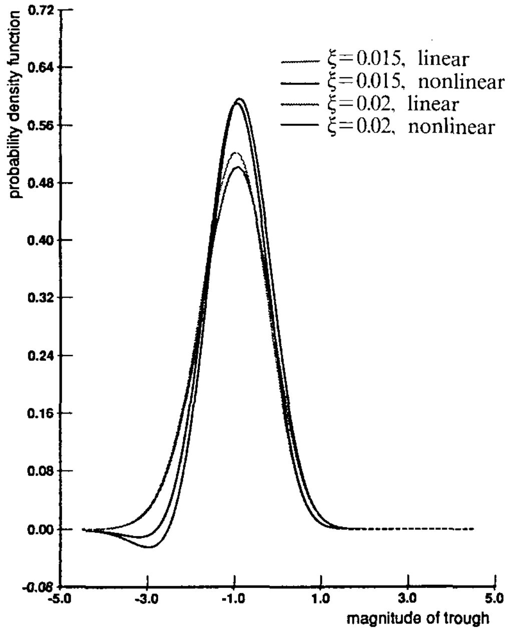

Fig. 5. Trough distribution.

6. TROUGH DISTRIBUTION

Inserting (40) into (4) and after some manipulation, the probability density function of troughs of nonlinear random waves, $f_{\zeta 0}(\zeta_0)$ is given by

$$f_{\zeta 0}(\zeta_0)=F_t(\zeta_0)/N_t \tag{60}$$

where

$$F_t(\zeta_0)=\int_0^{\infty}|\zeta_3|f_{\zeta_1\zeta_2\zeta_3}(\zeta_0,\ 0,\ \zeta_3)d\zeta_3 \tag{61}$$

$$=-\tilde{M}\int_{-\rho_1\zeta_0/\sqrt{1-\rho_1^2}}^{\infty}exp\left(-\frac{1}{2}z^2\right)dz+\tilde{N} \tag{62}$$

$$N_t=\frac{1}{2\pi}\left\{1+\varepsilon(1-\rho_2^2)\sqrt{2\pi}\right\} \tag{63}$$

In (62), $\tilde{M}$ and $\tilde{N}$ are given by (48) and (49), respectively. As $\varepsilon=0$, the trough distribution is reduced to

$$f_{\zeta 0}(\zeta_0)=\frac{1}{\sqrt{2\pi}}\left\{\rho_1\zeta_0\exp\left(-\frac{1}{2}\zeta_0^2\right)\cdot\int_{-\rho_1\zeta_0/\sqrt{1-\rho_1^2}}^{\infty}\exp\left(-\frac{1}{2}z^2\right)dz\times\sqrt{1-\rho_1^2}\cdot\exp\left(-\frac{1}{2}\frac{\zeta_0^2}{1-\rho_1^2}\right)\right\} \tag{64}$$

For a narrow band case, $\rho_1=-1$, (64) reduces to Rayleigh distribution as expected. The probability density function in (60) is plotted in Fig. 4 for $\xi=$ 0.015 and 0.02. In Fig. 5 the probability density function of trough of nonlinear random waves is plotted again with the trough distribution of linear waves for $\xi=0.015$ and 0.025. It is noted that for random process of moderate bandwidth, the troughs can reach above the mean water level and the trough distribution associated with nonlinear waves is more peaked than the linear counterpart and shifts its density mass toward positive value in an increasing manner as significant wave slope gets larger. It is also noted that the trough distribution in (60) gives negative value at the negative extreme trail of the function. This anamoly can be attibuted to the assumption made in the application of the perturbation methold. In Fig. 6, the trough distribution of nonlinear waves of narrow bandwidth and moderate bandwidth are plotted together for $\xi=0.015$.

7. CONCLUSION

Although a great deal of recent progress has been

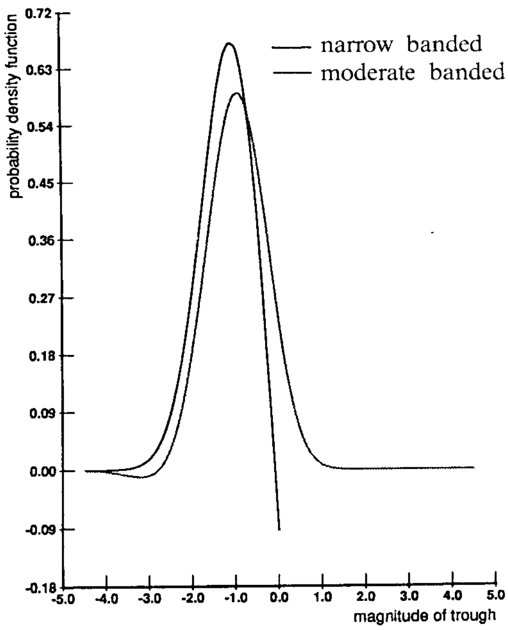

Fig. 6. Trough distribution (for $\xi=0.015$).

made on the theory of nonlinear waves, the complicated form of nonlinear random waves has made its application difficult. In particular, theoretical treatment of the statistical properties and distribution relevant to nonlinear wave field, such as crest and trough of wave elevation have not been found. In a case when the underlying frequency spectrum is narrow, the stochastic representation of a nonlinear sea surface is reduced to a familiar form in which each realization is an amplitude modulated second order Stokes wave. In contrast with the intricate complexity of the expression of nonlinear waves of finite bandwidth, such an approximation constitutes a simpler formulation to study numerically or analytically the nonlinear effects on the statistical description of wave properties. But considering the side band instability of Stokes wave, the narrow band assumption at the site away from the generating area is no longer valid. For waves of finite bandwidth, an approximate wave model proposed by Tung *et al.* (1989) is promising alternative from which the joint distribution of nonlinear wave elevation, its first and second derivatives can be obtained and the structure of which is simple enough so that statistical properties of such nonlinear random waves can be obtained. Based on this wave model, first, the joint distribution of wave elevation, its first and second derivatives were derived and the crest and trough distributions of nonlinear random waves were obtained. It was shown that as significant wave slope increases, these extreme distributions deviate from the linear counterpart in an increasing manner. The general character of this deviation is in the form of a spreading of the density mass toward the larger and smaller crests and smaller troughs, which is consistent with the vertically asymmetric properties of nonlinear waves which are known to have shallower troughs, and sharper and larger crests than the linear counterpart.

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