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Mach Reflection of Sinusoidally-Modulated Nonlinear Stokes Waves by a Thin Wedge

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Abstract

By using multiple-scale expansion techniques, the Mach reflection of sinusoidally-modulated nonlinear Stokes waves by a stationary thin wedge has been studied within the framework of potential theory. It is shown that the evolution of diffracted wave amplitude can be described by the Zakharov equation to the leading order and that it reduces to the cubic Schrödinger equation with an additional linear term in the case of stable modulations. Computations are made for the cubic Schrödinger equation for different values of nonlinear and dispersion parameters. Numerical results reflect the experimental findings in terms of the amplitude and width of generated stem waves. Based on the computations it is concluded that the nonlinearity dominates the wave field, while the dispersion does not significantly affect the wave evolution.

1. Introduction

It has long been observed in tank experiments that stem waves are generated in addition to reflected waves when solitary waves are incident to a vertical wall with an angle less than 45°[1, 2]. The crestline of the stem waves is almost perpendicular to the wall and the junction point between the stem and reflected waves are slightly apart from the wall. Particularly when the incident angle is less than 20°, no reflected waves appear leaving only the stem waves. Wiegel [3] claimed that the configuration is very similar to the Mach reflection of shock waves and hence coined the name. Nielsen [4], Berger & Kohlhase [5] observed the same phenomena in water of finite depth.

From these experiments, the stem waves are characterized as follows : The height of the stem waves first increases and then gradually decreases in the downstream. Its relative magnitude is proportional to the incident angle and inversely proportional to the incident wave amplitude. The stem width is wider for larger waves, and narrower for larger incident angle and also for greater waterdepth.

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Miles [6] clarified that the basic mechanism of the March reflection is the long-scale evolution of nonlinear waves. The long scale hereby is meant as the relative scale in comparison with those of carrier waves. It is well known, however, that nonlinear wave trains are likely to be unstable, when they are subject to any small side-band disturbances [7]. Therefore it is natural to raise the question about the stability of the Mach reflection, which is a very difficult task.

As a first step towards the task, it is aimed in this paper to investigate the nonlinear diffraction of stable wave trains.

2. Description of the Problem

Let us consider nonlinear waves incident to a long wedge, of which the angle is 2α. The frequency and the wave number of carrier waves are denoted by ω and k, respectively and its modulation frequency by Ω. Under the usual assumptions of potential flow, a velocity potential is introduced to describe the flow field in a right-handed Cartesian coordinate system as depicted in Fig.1. The problem may be formulated below :

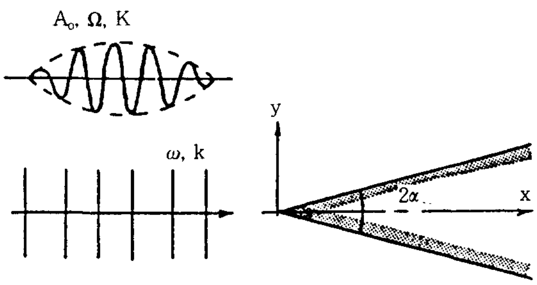


Fig. 1 Definition sketch

$\nabla^2\Phi=0$	$\infty<z<\zeta,$	(1)
$\Phi_z=\zeta_t+\zeta_x\Phi_x+\zeta_y\Phi_y$	$z=\zeta,$	(2)
$\Phi_t+1/2\lvert\nabla\Phi\rvert^2+gz=0$	$z=\zeta,$	(3)
$\Phi_n=0$	$y=\pm y_B(x),$	(4)
$\Phi_z\to 0$	$z\to-\infty,$	(5)
$\Phi=\Phi^I$	$x<0,$	(6)

where g corresponds to the gravitational acceleration, $\zeta(x, y, t)$ to the free surface, $\vec{n}$ to unit normal vector directed out of the fluid region, y_B to the surface equation of the wedge and Φ^I to the incident wave potential.

This boundary-value problem is too complicated to yield an exact solution. For an approximate solution, we need some simplifications. First of all, the wedge is assumed to be so thin that waves are diffracted only in forward direction. Thus the boundary condition, Eq.(6), can be justified. The wave steepness is as small as the thinness of the wedge, i.e.,

$$kA_o=\varepsilon\delta \qquad \delta=0(1) \qquad (7)$$

where A_0 stands for the typical wave amplitude.

Furthermore, the modulation frequency and the associated wave number are small compared to their counterparts of carrier waves in the following way :

$$\Omega/2\omega=\varepsilon\nu,\ \nu=0(1)\ ;\ K/2k=\varepsilon\mu,\ \mu=0(1) \qquad (8)$$

The amplitude dispersion of the Stokes waves is a typical nonlinear effect, of which the order of magnitude is $0(\varepsilon^3)$

$$\omega^2=gk[1+(kA_o)^2] \qquad (9)$$

It implies that the nonlinear dispersion becomes effective only when the carrier wave travels a long distance of $0(k^{-3}Ao^{-2})$. Accordingly it is reasonable to take the dimensions of the wedge.

$$kL=0(\varepsilon^{-2}) \quad \text{and} \quad kB=0(\varepsilon^{-1}) \qquad (10)$$

where L and B are the length and the beam of the wedge, respectively. It is of interest to note that Eq.(10) is nothing but the thin ship assumption, i.e. $B/L=0(\varepsilon)$. It is recalled that the depth attenuation of wave motions in deep water is exponential. Thus the primary picture of the diffracted waves by a thin ship may be captured on the free surface, which would not differ significantly from the diffraction by a vertical wall instead of a thin ship. Furthermore only the forward diffraction has been assumed here. Based on these reasonings, we may conjecture that the present analysis can be applied for the nonlinear bow wave diffraction of a thin ship.

3. Evolution of Diffracted Wave Amplitude

Either multiple-scale expansion technique or inverse scattering method is employed for nonlinear wave problems [8, 9]. In our case, it is pertinent to take the former method because the temporal and spatial scales of variables involved in the problem are vastly different. We may classify them into three groups ; the spatial and the temporal scales associated with carrier waves (x, t), those associated with modulation (K^{-1}, Ω^{-1}) and the length scale of the wedge(L). The orders of the relative magnitudes of these variables are $0(1)$, $0(\varepsilon^{-1})$, $0(\varepsilon^{-2})$, respectively. To deal with these variables of different scales, the long-scale variables are introduced.

$$x_1=\varepsilon x,\ \varepsilon^2 x,\ y_1=\varepsilon y,\ t_1=\varepsilon t. \qquad (11)$$

The velocity potential and the free surface are first expanded asymptotically with respect to ε and then further expanded into Fourier series with respect to t.

$$\Phi(x,y,z,t)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n} e^{im\varphi}\,\phi_{nm}(x_1,\ x_2,\ y_1,\ t_1) \qquad (12)$$

$$\zeta(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n} e^{im\varphi}\,\eta_{nm}(x_1,\ x_2,\ t_1) \qquad (13)$$

where $\varphi(=kx-\theta t)$ is the phase function of carrier waves.

It is understood that only the real parts are to be taken. Since the surface elevation must be real, the following relations hold

$$\phi_{n-m}=\phi_{nm}^*,\ \eta_{n-m}=\eta_{nm}^*,$$

where * denotes the complex conjugate.

Recalling the modulation and the amplitude dispersion, the incident wave potential is given by

$$\Phi^I(x,z,t)=(-ig/\omega)\exp(kz+i\varphi)\ A(x_1,\ x_2,\ t_1) \qquad (14)$$

If sinusoidal modulation is assumed, the amplitude function takes the form

$$\begin{aligned} A&=A_o \exp i(Kx-\Omega t)-k^3 Ao^2 x) \qquad (15)\\ &=A_o \exp i(2\mu k x_1-2\nu\omega k t_1-\delta^2 k x_2) \end{aligned}$$

Since the solution procedure in general is given in the comprehensive book of Mei [9, pp.607−618], detailed derivations are omitted here. The first-order solution is well known and trivial sofar. The second-order solution requires the conservation of wave action

$$\partial A/\partial t_1+C_g\,\partial A/\partial x_1=0 \quad (C_g=\text{group velocity}) \qquad (16)$$

If Eq.(15) is put into it, we obtain $2\nu=\mu$. The body surface is expressed in terms of the long-scale variable

$$y_B=kB\ Y_B(x_2) \qquad (17)$$

then the kinematic boundary condition turns out to be

$$\partial\Phi/\partial y_1=[(\partial/\partial x+\varepsilon^2\,\partial/\partial x_2)\,\Phi]\,\partial Y_B/\partial x_2 \qquad (18)$$

where Eq.(10) has been used.

It is clear that the linear solution must be

$$\Phi=(-ig/\omega)\exp(kz+i\varphi)\ A(x_1,\ x_2,\ y_1,\ t_1) \qquad (19)$$

By substituting it into Eq.(18) and collecting terms of lending order, we have

$$\partial A/\partial y_1 = ikA\,\partial Y_B/\partial x_2 \quad y_1 = Y_B(x_2) \tag{20}$$

The evolution of the amplitude is obtained at the next order,

$$C_g\,\partial A/\partial x_2 + i\omega/8k^2(\partial^2 A/\partial x_1^2 - 2\partial^2 A/\partial y_1^2) + (i/2)\omega k^2|A|^2 A = 0 \tag{21}$$

This equation was derived for the first time by Zakharov [10]. It is to note that the general form of the solution contains an integral constant which physically represents a current-like flow. In the above equation, this term has been simply discarded, because the current is small in deep water.

It is more convenient to introduce the following nondimensional variables.

$$A = A/Ao,\ X_1 = kx_1,\ X_2 = kx_2,\ Y = ky_1,\ T = \omega t_1 \tag{22}$$

In terms of these variables, Eq.(21) is rewritten.

$$\partial A/\partial X_2 + (i/4)(\partial^2 A/\partial x_1{}^2 - 2\partial^2 A/\partial Y^2) + i\delta^2|A|^2 A = 0 \tag{23}$$

If it can be assumed that the diffracted waves are also modulated sinusoidally, then the amplitude function of the diffracted waves can be supposed to be

$$A(x_1, x_2, y_1, t_1) = A(x_2, y_1)\exp i(2\mu k x_1 - \mu\omega t_1) \tag{24}$$

By substituting it into Eq.(23), we finally obtain the cubic Schrödinger equation.

$$\partial\tilde{A}/\partial X_2 - (i/2)\partial^2\tilde{A}/\partial Y^2 + i(-\mu^2 + \delta^2|\tilde{A}|^2)\tilde{A} = 0 \tag{25}$$

where the amplitude is normalized by Ao.

The first term represents the evolution of the wave amplitude with respect to the longitudinal distance, X_2, and the second term corresponds to its lateral diffusion, while the third term contains both the dispersion and the nonlinearity. If the incident waves are uniform, the last term reflects only the nonlinearity [11]. Eq.(25) may be interpreted as the equation of motion for an oscillator of two-degrees of freedom oscillator with a nonlinear spring, which belongs to a class of Duffing's equations [12].

4. Numerical Result and Discussion

Based on the instability theory of Benjamin & Feir [7], Longuet-Higgins [13] investigated the instability problem by numerical method. In contrast to the previous result, he found that the borderline between stable and unstable zones is not a simple straightline but a curve on the kAo $(=\varepsilon\delta) - K/2k(=\varepsilon\mu)$ plane. He also found that Stokes waves are always unstable at $\varepsilon\delta = 0.41$ for all μ. However, as clearly stated at the beginning of the paper, we restrict our attention her only to stable conditions. Our primary concern is the effect of the wedge angle and the nonlinearity.

In the present computations, the Crank-Nicholson algorithm has been utilized for X derivatives and the centered difference is taken for Y derivatives. To limit the dimension of the computation domian, the following boundary condition is imposed at the boundary, where $Y >> 1$,

$$\partial A/\partial Y = 0,\ A = \exp i(2\mu X_1 - \mu T - \delta^2 X_2) \tag{26}$$

The wedge is expressed by

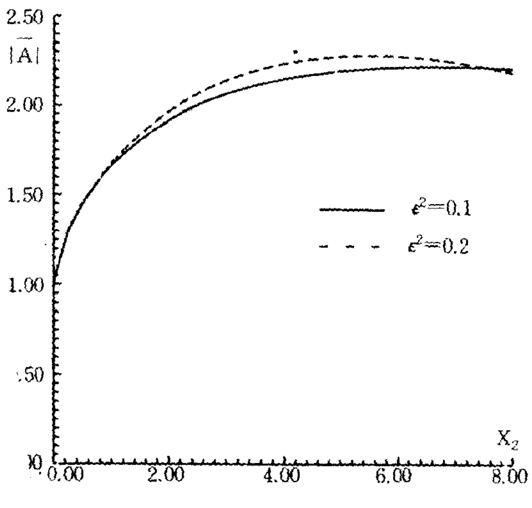
$$y = x\tan\alpha,\ \tan\alpha = \varepsilon \tag{27}$$

and accordingly the kinematic body boundary condition is simplified to

$$\partial A/\partial Y = iA \qquad Y = X_2 \tag{28}$$

Specifically the computations are made for the two wedges, of which the half angles are 17.55° ($\varepsilon^2=0.1$) and 24.09° ($\varepsilon^2=0.2$). Three wave steepnesses are considered, i.e. kAo=0.05, 0.1, 0.3. The corresponding values of δ are 0.1582, 0.3163, 0.9489 for $\alpha=17.55°$ ($\varepsilon^2=0.1$) and 0.1118, 0.2236, 0.6708 for $\alpha=24.09°$ ($\varepsilon^2=0.2$). Two modulations are taken, i.e. K/2k=0.1 and 0.3. According to the result of Longuet-Higgins[13], these modulations are stable except the cases of K/2k=0.1 for kAo=0.1 and 0.3. The nondimensional length of one modulation is $X_1=\pi/2\mu$ and it takes the nondimensional time of $To=\pi/\mu$ for carrier waves to propagate over this distance.

Fig.2 shows the variation of diffracted wave amplitude along the wedge for kAo=0.1 and K/2k=0.1. The solid line is designated by $\varepsilon^2=0.1$ corresponds to the wedge of half angle $\alpha=17.55°$, while the dotted line designated by $\varepsilon^2=0.2$ to the wedge of $\alpha=24.09°$. The ordinate is the normalized wave amplitude and the abscissa is the nondimensional long-scale length X_2. It is



2 Variation of wave amplitude along the wedge for kAo=0.1 and K/ 2k=0.1

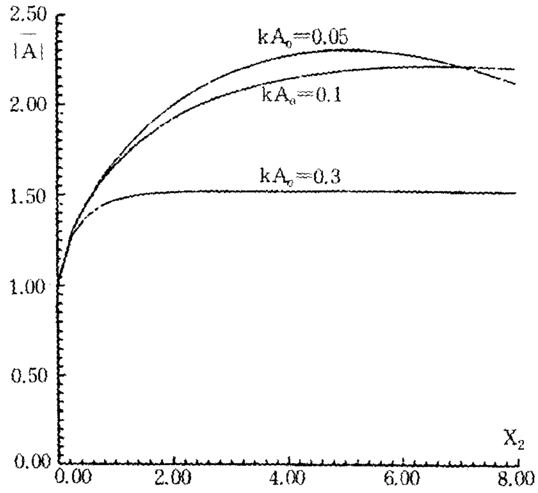


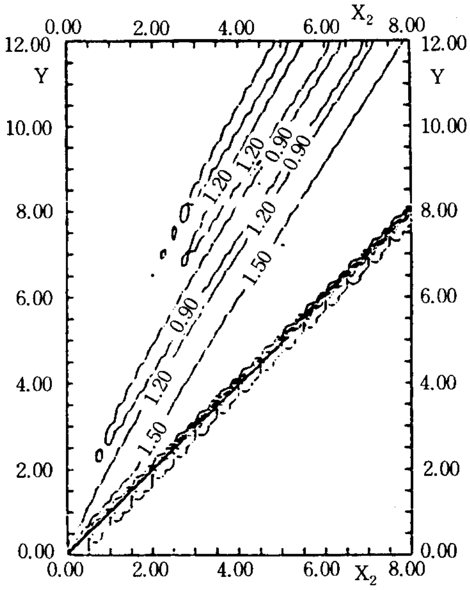
Fig. 3 Variation of wave amplitude along the wedge for different wave slopes with $\varepsilon^2=0.1$ and K/ 2k=0.1

observed in the figure that the amplitude of diffracted waves starts to increase monotonically and then decreases very slowly. The magnitude is slightly greater for the wider wedge.

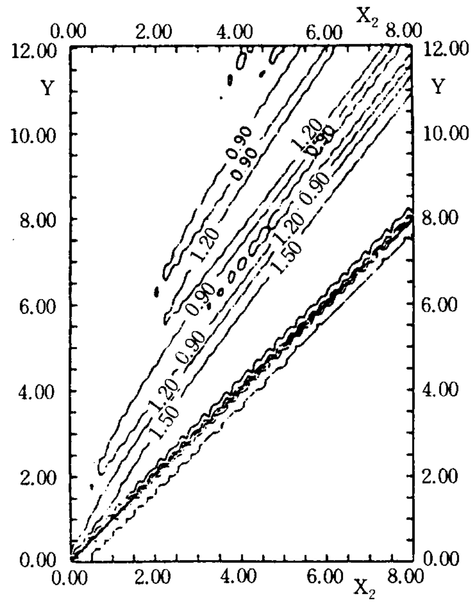
Variations of wave amplitude due to the incident wave steepness are illustrated in Fig.3. Here the half angle is $\alpha=17.55°$ and the modulation ratio is K/2k=0.1. The relative wave amplitude decreases considerably for the steepest waves, kAo=0.3.

Fig.4 shows the snapshots of wave field for kAo=0.3 and K/2k=0.1, in which the ordinate represents the lateral coordinate Y. Attention should be paid to the different scales of the lateral length $Y(=k\varepsilon y)$ from that of the horizontal length $X_2(=k\varepsilon^2 x)$. The numbers denote the dimensionless wave amplitude. The overall features for two different wedge angles are similar each other. Stem waves are observed near the wedge and its width increases almost linearly downstream. The stem angle is measured to be 7.2° for the narrower wedge and 7.0° for the wider wedge. Although the difference is small, its trend is in the direction to support the experiments.

Lastly the lateral variations of wave amplitude

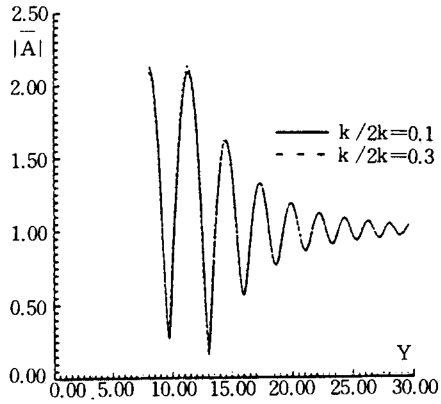


(a) $\varepsilon^2=0.1$

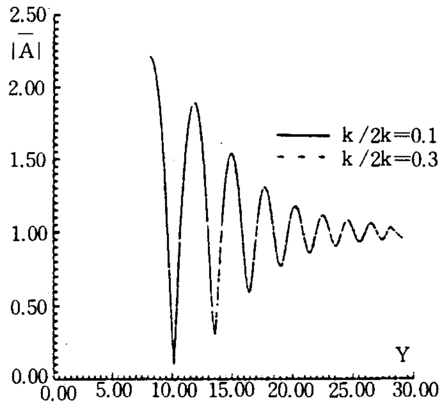


(b) $\varepsilon^2=0.2$

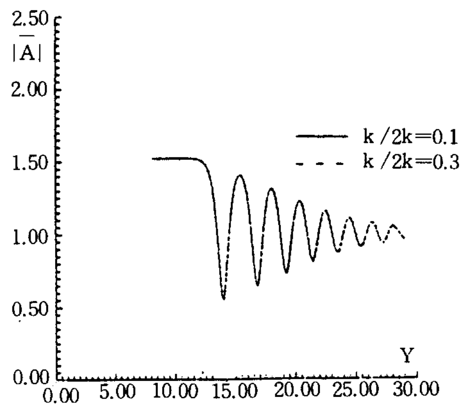
Fig. 4 Contour of diffracted waves along the wedge for kAo=0.3 and K/ 2k=0.1



(a) $kA_0=0.05$



(b) $kA_0=0.1$



(c) $kA_0=0.3$

Fig. 5 Lateral variation of wave ampltude at X2=8 different modulation ratios with $\varepsilon_z=0.1$ and K/ 2k=0.3

at the location of $X_2=8.0$ are depicted in Fig.5. The half wedge angle is $\alpha=17.55°$ and the wave steepnesses are (a) kAo=0.05, (b) kAo=0.1 and (c) kAo=0.3. The solid and dotted lines represent the modulation ratios for K /2k=0.1 and 0.3 respectively. Stem waves are observed only for the case of steep waves, (c) kAo=0.3. In the case of (a) kAo=0.05, the second crest from the wall has almost the same magnitude as that at the wall. Generally speaking it is hard to recognize any differences between two cases in the figures. In implies that the wave evolution is not significantly affected by the dipsersion.

5. Conclusions

In this paper, a study has been made on the diffraction of sinusoidally-modulated nonlinear waves by using the multiple expansion method. Under the assumption of forward diffractions, it is shown that the evolution of the diffracted wave amplitude can be described by the Zahkarov equation, which further reduces to the cubic Schrödinger equation when the envelope of carrier waves is modulated sinusoidally.

Numerical results support the experimental findings in general. The cubic Schrödinger equation contains the dispersion term, but it is numerically confirmed that its effect is negligibly small. Meanwhile the nonlinearity affects the wave evolution significantly.

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