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Random Upper Functions for Levy Processes†

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ABSTRACT

Let $\{X(t) : t \geq 0\}$ be a real-valued stochastic process with stationary independent increments. In this paper, under the condition of stochastic compactness, we obtain appropriate function $\alpha(t)$ and random function $\beta(t)$ such that for some positive finite constant C, $\limsup\{X(t) - \alpha(t)\}/\beta(t) = C$ a.s. both as t tends to zero and infinity.

1. INTRODUCTION

Let $\{X(t) : t \geq 0\}$ be a real-valued stochastic process with stationary independent increments whose characteristic function is given by

$$Eexp(iuX(t)) = exp(tg(u)),$$

where

$$g(u) = ibu + \int (e^{iux} - 1 - iux/(1+x^2))\, d\nu(x)$$

and ν is a Levy measure on $R - \{0\}$ satisfying

$$\int (x^2 \wedge 1)\, d\nu(x) < \infty.$$

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For $a > 0$, we define

$$\begin{aligned} G(a) &= \int_{|x|>a} d\nu(x), \\ K(a) &= a^{-2} \int_{|x|\le a} x^2 \, d\nu(x), \\ M(a) &= b + \int_{|x|\le a} x^3/(1+x^2)\, d\nu(x) - \int_{|x|>a} x/(1+x^2)\, d\nu(x), \\ Q(a) &= G(a) + K(a). \end{aligned} \tag{1.1}$$

It follows immediately that Q is positive, continuous, decreasing and zero at infinity. Also it is clear that $a^2 Q(a) = \int (x^2 \wedge a^2)\, d\nu(x)$ is nondecreasing.
Our basic assumption throughout this paper will be

$$\lim sup\ G(x)/K(x) < \infty \tag{1.2}$$

both as $x \to 0$ and $x \to \infty$. It is well konwn that this analytic condition (1.2) is equivalent to stochastic compactness for $\{X(t) : t \ge 0\}$ (See Theorem 2 of Ahn and Wee(1985)). Thus it seems natural that we ask whether under (1.2) there exist appropriate functions $\alpha(t)$ and $\beta(t)$ such that for some nonzero finite constant C,

$$\lim sup\ (X(t) - \alpha(t))/\beta(t) = C \quad a.s. \tag{1.3}$$

both as $t \to 0$ and $t \to \infty$.

The classical result in this area is that $\beta(t)$ is a positive nondecreasing function on $[0, \infty]$ with $\beta(0) = 0$, $\beta(\infty) = \infty$.
For example, if $X(t)$ is a Brownian motion, (1.3) holds with $\beta(t) = (tlog|logt|)^{1/2}$. Also for a stable process of exponent α having only negative jumps, it was proved that (1.3) is valid with $\beta(t) = t^{1/\alpha}(log|logt|)^{1-1/\alpha}$ when $\alpha \ne 1$, and $\beta(t) = t|logt|$ when $\alpha = 1$.
For the case of general Levy processes having only negative jumps, Kim and Wee(1991) obtained the upper functions under some extra conditions.

The purpose of this paper is to consider a similar problem when $\beta(t)$ is a random function. If $X(t)$ is a Brownian motion and $T(t)$ is arbitrary subordinator with a positive drift term bt, then (1.3) holds as $t \to 0$ with $\beta(t) = (T(t)log|logt|)^{1/2}$ since $T(t)/t \to b$ $a.s.$ as $t \to 0$. This idea shows that our problem can be solved with $\beta(t) = (T(t)log|logt|)^{1/2}$ if we find an appropriate subordinator $T(t)$. It turns out that the subordinator $T(t)$ defined by

$$T(t) = \lim_{\epsilon \to 0} \sum_{s \leq t} (X(s) - X(s^-))^2 I\{|X(s) - X(s^-)| > \epsilon\} \tag{1.4}$$

is suitable for our purpose. To describe our results, we first note that a Gaussian component is excluded since otherwise the process behaves near the origin as if the non-Gaussian part were zero. And as usual, we assume that $X(0) = 0$ and that we are dealing with a version which has almost all sample functions right continuous and having left limits. Then our main result is as follows:

Theorem 1.

(1) Suppose $\limsup_{x \to 0} G(x)/K(x) < \infty$.

(a) If we define $b_\lambda(t)$by $Q(b_\lambda(t)) = log|logt|/\lambda t$ for $\lambda > 0$,

then for all $\lambda > 0$,

$$\limsup_{t \to 0} \{X(t) - tM(b_\lambda(t))\}/(T(t)log|logt|)^{1/2} = C(\lambda) \ a.s.$$

where $C(\lambda)$ is a finite constant.

(b) If λ is sufficiently large, then $C(\lambda)$ is strictly positive.

(2) The similar conclusions hold if $t \to 0$ and $x \to 0$ are replaced by

$t \to \infty$ and $x \to \infty$ respectively.

Corollary. If $X(t)$ is a stable process of exponent α , then the above theorem is valid with $T(t)$ a stable subordinator of exponent $\alpha/2$.

An analogous problem for sums of i.i.d. random variables was solved by Griffin and Kuelbs(1989) under the condition of stochastic compactness, which generalizes the classical LIL. In fact, our present work was motivated by the results of Griffin and Kuelbs(1989) and we obtain the continuous analogue of their results by replacing distribution function with Levy measure.

Now the organization of this paper will be described briefly. Section II consists of the basic facts and some lemmas which will be used in later sections.

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In Section III, we prove the part (a) of (1) in Theorem, and the proof of the part (b) is given in Section IV. Here we omit the proof for the case of $t \to \infty$ since it follows from similar arguments.

Throughout this paper, the following notation will be used very frequently;

(1) I(A) is the indicator function of A.

(2) As $x \to 0$ and $x \to \infty$, $f(x) \sim g(x)$ iff $f(x)/g(x) \to 1$.

(3) $L_2 t = log|logt|$.

2. PRELIMINARIES

In this section, we give some basic facts and obtain several lemmas which will be used in later sections. We start with the decomposition of $X(t)$ as follows; for $k = 1, 2$, let

$$Eexp(iuX_k(t : a)) = exp(tg_k(u)),$$

where $g_1(u) = \int_{|x| \leq a}(e^{iux} - 1 - iux)\, d\nu(x)$ and $g_2(u) = \int_{|x|>a}(e^{iux} - 1)\, d\nu(x)$.

Then $X(t) = X_1(t : a) + X_2(t : a) + tM(a)$, where the processes are independent. It follows easily that by differentiating the characteristic function of $X_1(t : a)$,

$$EX_1(t : a) = 0, \ \text{Var} X_1(t : a) = ta^2 K(a).$$

We note that $X_2(t : a)$ is a compound Poisson process with parameter $G(a)$, and so $P\{X_2(s : a) = 0 \text{ forall } s \leq t\} = e^{-tG(a)}$.

Lemma 2.1. For any $s > 0$ and $v > 0$,

$$P\{X_1(t : a) \geq 2^{-1} v e^v ta K(a) + sav^{-1}\} \leq e^{-s}.$$

Proof. See Lemma 2.1 of Kim and Wee(1988).

Lemma 2.2. If $tK(a)$ is sufficiently large, then there exist $\xi_1,\ \xi_2 \in (0,1)$ such that

$$P\{X_1(t:a) \geq \xi_1 taK(a)\} \geq exp(-\xi_2 tK(a)).$$

Proof. See Lemma 2.2 of Kim and Wee(1988).

Now, if

$$\limsup_{x\to 0} G(x)/K(x) < \theta < \infty, \tag{2.1}$$

then by Lemma 4.2 of Wee (1988), for x sufficiently small,

$$x^p Q(x) \text{ is strictly decreasing,} \tag{2.2}$$

where $p = 2/(1+\theta)$ is used throughout the remainder of this paper.

For $\lambda > 0$, we define

$$\begin{aligned} Q(a(t)) &= 1/t, \\ b_\lambda(t) &= a(\lambda t/L_2 t), \\ \alpha_\lambda(t) &= b_\lambda(t) L_2(t), \\ \beta_\lambda(t) &= b_\lambda^2(t) L_2 t. \end{aligned} \tag{2.3}$$

Then $b_\lambda(t)$ and $\beta_\lambda(t)$ increase for small t.

Lemma 2.3. If (2.1) holds, then for $y \leq x$ and all x sufficiently small,

$$(y/x)^{1/p} \leq a(y)/a(x) \leq 1. \tag{2.4}$$

Proof. By (2.2), for $y \leq x$ and x is sufficiently small,

$$a^p(y) = ya^p(y)Q(a(y)) \geq ya^p(x)Q(a(x)) = (y/x)a^p(x),$$

which gives the desired result.

Lemma 2.4. Let (2.1) hold, $t_k = \rho^{-k}$ with $\rho > 1$ and p be as in (2.2). Then for each $\lambda > 0$,

$$\rho^{-1/p} \leq \lim inf_k b_\lambda(t_{k+1})/b_\lambda(t_k) = \lim inf_k \alpha_\lambda(t_{k+1})/\alpha_\lambda(t_k) \leq 1, \tag{2.5}$$

and

$$\lim sup_k t_k \int |x| I\{b_\lambda(t_{k+1}) < |x| \leq b_\lambda(t_k)\}\, d\nu(x)/\alpha_\lambda(t_{k+1}) \leq \lambda^{-1}\rho^{1+1/p}. \tag{2.6}$$

Proof. (2.5) follows immediately from (2.4). To verify (2.6), apply (2.4) to obtain

$$\begin{aligned}
&\limsup_k t_k \int |x| I\{b_\lambda(t_{k+1}) < |x| \leq b_\lambda(t_k)\}\, d\nu(x)/\alpha_\lambda(t_{k+1}) \\
&\quad \leq \limsup_k t_k b_\lambda(t_k) G(b_\lambda(t_{k+1}))/\alpha_\lambda(t_{k+1}) \\
&\quad \leq \limsup t_k Q(b_\lambda(t_{k+1})) b_\lambda(t_k)/\alpha_\lambda(t_{k+1}) \\
&\quad \leq \lambda^{-1}\rho^{1+1/p}.
\end{aligned}$$

Thus the lemma is proved.

Lemma 2.5. Let $t_k = \rho^{-k}$ with $\rho > 1$, $I(k) = (t_{k+1}, t_k]$ and let

$$J_\lambda(t) = \text{the number of } s \leq t \quad \text{for which } |X(s) - X(s^-)| > b_\lambda(t_k) \tag{2.7}$$

for $t \in I(k)$. Then for each $\lambda > 0$,

$$\limsup_{t \to 0} J_\lambda(t)/L_2 t \leq \eta_1(\lambda), \tag{2.8}$$

where

$$\eta_1(\lambda) = inf\{\xi > \lambda^{-1} : \xi(log(\xi\lambda) - 1) + \lambda^{-1} > 1\} \tag{2.9}$$

Proof. First we note that $J_\lambda(t)$ is a Poisson process with rate $G(b_\lambda(t_k))$. Thus for $\xi > 0$, $u > 0$,

$$\begin{aligned} P\{\sup_{t\in I(k)} J_\lambda(t)/L_2t > \xi\} &\leq P\{J_\lambda(t_k) > \xi L_2 t_k\} \\ &\leq E[exp(uJ_\lambda(t_k))]exp(-u\xi L_2 t_k) \\ &= exp[t_k G(b_\lambda(t_k))(e^u - 1)]exp(-u\xi L_2 t_k) \\ &\leq exp[t_k Q(b_\lambda(t_k))(e^u - 1) - u\xi L_2 t_k] \\ &= exp[-L_2 t_k(u\xi - \lambda^{-1}e^u + \lambda^{-1})]. \end{aligned}$$

Now assume $\xi > \eta_1(\lambda)$ and set $u = log \xi\lambda$. Then $u > 0$ and hence

$$P\{\sup_{t\in I(k)} J_\lambda(t)/L_2t > \xi\} \leq exp[-L_2 t_k(\xi(log(\xi\lambda) - 1) + \lambda^{-1}].$$

Since $\xi(log(\xi\lambda) - 1) + \lambda^{-1}$ is an increasing function in ξ for $\xi > \lambda^{-1}$, we conclude that for $\xi > \eta_1(\lambda)$, $\xi(log(\xi\lambda) - 1) + \lambda^{-1} > 1$ which implies

$$\sum_k P\{\sup_{t\in I(k)} J_\lambda(t)/L_2t > \xi\} < \infty.$$

Thus the lemma is proved.

We now introduce an increasing Levy process $T(t)$ defined by

$$T(t) = \lim_{\epsilon\to 0}\sum_{s\leq t}(X(s) - X(s^-))^2 I\{|X(s) - X(s^-)| > \epsilon\}, \qquad (2.10)$$

which has the following characteristic function;

$$Eexp(iuT(t)) = exp\{t\int(e^{iux} - 1)\,d\mu(x)\}, \qquad (2.11)$$

where μ is a Borel measure on $(0,\infty)$ defined by $\mu(E) = \nu(E^{1/2}) + \nu(-E^{1/2})$. (We adopt the notation; $E^{1/2} = \{x^{1/2} : x \in E\}$.)

Now let

$$Eexp(iuT_\lambda(t)) = exp(t\int_{x\leq b_\lambda^2(t)}(e^{iux} - 1)\,d\mu(x)). \qquad (2.12)$$

Then

$$\begin{aligned} ET_\lambda(t) &= t\int_{x\le b_\lambda^2(t)} x\, d\mu(x) \\ &= t\int_{|x|\le b_\lambda(t)} x^2\, d\nu(x). \end{aligned}$$

Thus if (2.1) holds, then for t sufficiently small

$$\beta_\lambda(t)/(\lambda(1+\theta)) \le ET_\lambda(t) \le \beta_\lambda(t)/\lambda. \tag{2.13}$$

Lemma 2.6. Suppose that (2.1) holds, and for $\lambda > 0$ let

$$\eta_2(\lambda) = 2^{-1}\lambda^{1/2}e^{\lambda^{1/2}} + 2\lambda^{1/2}. \tag{2.14}$$

Then for each $\lambda > 0$

$$\limsup_{t\to 0} T_\lambda(t)/\beta_\lambda(t) \le (1+\eta_2(\lambda))\lambda^{-1}. \tag{2.15}$$

Furthermore, if $\lambda > 0$ is sufficiently small such that $\eta_2(\lambda) < (1+\theta)^{-1}$, then

$$\liminf_{t\to 0} T_\lambda(t)/\beta_\lambda(t) \ge [(1+\theta)^{-1} - \eta_2(\lambda)]\lambda^{-1}. \tag{2.16}$$

Finally, if $\lambda_0 > 0$ is such that $\eta_2(\lambda_0) < (1+\theta)^{-1}$, then for all $\lambda \ge \lambda_0$,

$$\liminf_{t\to 0} T(t)/\beta_\lambda(t) \ge [(1+\theta)^{-1} - \eta_2(\lambda_0)](\lambda_0/\lambda)^{2/p}\lambda_0^{-1} > 0. \tag{2.17}$$

Proof. Applying Lemma 2.1 with $a = b_\lambda^2(t)$, $s = 2L_2t$ and $v > 0$, we obtain

$$\begin{aligned} &P\{|T_\lambda(t) - ET_\lambda(t)| > 2^{-1}ve^v tb_\lambda^2(t)K_\mu(b_\lambda^2(t)) + 2v^{-1}b_\lambda^2(t)L_2t\} \\ &\quad\le 2exp(-2L_2t), \end{aligned} \tag{2.18}$$

where K_μ is defined as in (1.1) by replacing ν with μ.

Setting $v = \lambda^{1/2}$ and noting that

$$\begin{aligned} K_\mu(x^2) &= x^{-4}\int_{y\le x^2} y^2\, d\mu(y) \\ &= x^{-4}\int_{|y|\le x} y^4\, d\nu(y) \\ &\le K(x), \end{aligned}$$

we obtain

$$P\{|T_\lambda(t) - ET_\lambda(t)| > \eta_2(\lambda)\beta_\lambda(t)/\lambda\} \le 2exp(-2L_2 t). \tag{2.19}$$

To verify (2.15), let $t_k = \rho^{-k}$ with $\rho > 1$ and $I(k) = (t_{k+1}, t_k]$.
If $\xi\rho^{-2/p} - 1 > \eta_2(\lambda)$, then by (2.5), (2.13) and (2.19), for each $\epsilon > 0$ and k sufficiently large,

$$\begin{aligned} &P\{\sup_{t\in I(k)} T_\lambda(t)/\beta_\lambda(t) > (\xi+\epsilon)/\lambda\} \\ &\quad\le P\{T_\lambda(t_k) > (\xi+\epsilon)\beta_\lambda(t_{k+1})/\lambda\} \\ &\quad\le P\{T_\lambda(t_k) - ET_\lambda(t_k) > \beta_\lambda(t_k)[(\xi+\epsilon)\beta_\lambda(t_{k+1})\beta_\lambda^{-1}(t_k) - 1]/\lambda\} \\ &\quad\le P\{|T_\lambda(t_k) - ET_\lambda(t_k)| > \beta_\lambda(t_k)(\xi\rho^{-2/p} - 1)/\lambda\} \\ &\quad\le 2exp(-2L_2 t_k). \end{aligned} \tag{2.20}$$

By the Borel-Cantelli lemma, (2.20) implies (2.15) since $\rho > 1$, $\xi > \rho^{2/p}(1 + \eta_2(\lambda))$ and $\epsilon > 0$ are arbitrary in (2.20).

To verify (2.16) and (2.17), let $\lambda > 0$ be sufficiently small such that $\eta_2(\lambda) < (1+\theta)^{-1}$. Fix $\rho > 1$ and $\xi > 0$ such that

$$(1+\theta)^{-1} - \xi\rho^{2/p} > \eta_2(\lambda). \tag{2.21}$$

Then by (2.5), (2.13) and (2.19), for any $\epsilon \in (0,\ \xi)$ and all k sufficiently large,

$$\begin{aligned} &P\{\inf_{t\in I(k)} T_\lambda(t)/\beta_\lambda(t) < (\xi-\epsilon)/\lambda\} \\ &\le P\{T_\lambda(t_{k+1}) < (\xi-\epsilon)\beta_\lambda(t_k)/\lambda\} \\ &\le P\{T_\lambda(t_{k+1}) - ET_\lambda(t_{k+1}) < \beta_\lambda(t_{k+1})[(\xi-\epsilon)\beta_\lambda(t_k)\beta^{-1}(t_{k+1}) - (1+\theta)^{-1}]/\lambda\} \\ &\le P\{|T_\lambda(t_{k+1}) - ET_\lambda(t_{k+1})| > \beta_\lambda(t_{k+1})[(1+\theta)^{-1} - \xi\rho^{2/p}/\lambda\} \\ &\le 2exp(-2L_2 t_{k+1}). \end{aligned} \tag{2.22}$$

The Borel-Cantelli lemma and (2.22) now imply (2.16) since $\rho > 1$,

$0 < \xi < [(1+\theta)^{-1} - \eta_2(\lambda)]\rho^{-2/p}$ and $\epsilon > 0$ are arbitrary in (2.21) and (2.22).

Finally, fix $\lambda_0 > 0$ so that $\eta_2(\lambda_0) < (1+\theta)^{-1}$ and let $\lambda \geq \lambda_0$.
Then by (2.16) and (2.4),

$$\begin{aligned}
&\liminf_{t\to 0} T(t)/\beta_\lambda(t) \geq \liminf_{t\to 0}\{T_{\lambda_0}(t)/\beta_{\lambda_0}(t)\}\{\beta_{\lambda_0}(t)/\beta_\lambda(t)\} \\
&\geq [(1+\theta)^{-1} - \eta_2(\lambda_0)]\lambda_0^{-1} \liminf_{t\to 0} b^2_{\lambda_0}(t)/b^2_\lambda(t) \\
&\geq [(1+\theta)^{-1} - \eta_2(\lambda_0)](\lambda_0/\lambda)^{2/p}\lambda_0^{-1}.
\end{aligned}$$

3. UPPER BOUND

In this section, we prove the part (a) of (1) in Theorem.
To do this, we adopt the following notation;

$$X_{1,\lambda}(t:r) = X_1(t:b_\lambda(r)), \quad X_{2,\lambda}(t:r) = X_2(t:b_\lambda(r)),$$

and

$$Eexp(iuT_\lambda(t:r)) = exp(t\int_{x\leq b^2_\lambda(r)} (e^{iux}-1)\, d\mu(x)).$$

In addition, for $\rho > 1$, set $t_k = \rho^{-k}$ and $I(k) = (t_{k+1}, t_k]$.

For $t \in I(k)$, we define

$$X_{1,\lambda}(t) = X_1(t:b_\lambda(t_k)), \quad X_{2,\lambda}(t) = X_2(t:b_\lambda(t_k)).$$

We start with the following well-known lemma.

Lemma 3.1. (Skorokhod) Let $\{W_n\}$ be a sequence of independent random variables and $S_n = \sum_{k=1}^n W_k$. Suppose that

$$P\{S_n - S_k \geq -\xi\} \geq C > 0 \quad \text{for all } k \leq n.$$

Then

$$P\{\max_{k\leq n} S_k \geq \lambda + \xi\} \leq C^{-1} P\{S_n \geq \lambda\}.$$

Proof. See Gihman and Skorokhod(1975) P.324.

Lemma 3.2. If (2.1) holds, then for each $\lambda > 0, \ \rho > 1$,

$$\limsup_{t \to 0} |X_{1,\lambda}(t)|/\alpha_\lambda(t) = C_1(\lambda, \rho) \ \ a.s., \tag{3.1}$$

where $C_1(\lambda, \rho)$ is a finite constant such that

$$C_1(\lambda, \rho) \le \rho^{1/p}\lambda^{-1/2}(2^{-1}e^{\lambda^{1/2}} + 1). \tag{3.2}$$

Proof. By zero-one law, $C_1(\lambda, \rho)$ is a constant. Thus it suffices to show that $C_1(\lambda, \rho)$ is finite. To show this, we proceed as follows. For each $\epsilon > 0$, Chebyshev's inequality and (2.5) imply

$$\begin{aligned}&\limsup_k \sup_{t\in I(k)} P\{|X_{1,\lambda}(t_k) - X_{1,\lambda}(t)| > \epsilon\alpha_\lambda(t_{k+1})\}\\ &\quad \le \limsup_k t_k b_\lambda^2(t_k)K(b_\lambda(t_k))\rho^{2/p}/(\epsilon b_\lambda(t_k)L_2 t_k)^2\\ &\quad \le \limsup_k \rho^{2/p}/\lambda\epsilon^2 L_2 t_k = 0.\end{aligned}$$

Hence by Lemma 3.1 and (2.5), for each $\delta > \epsilon > 0$ and all k sufficiently large,

$$\begin{aligned}&P\{\sup_{t\in I(k)} |X_{1,\lambda}(t)| > \delta\alpha_\lambda(t_{k+1})\}\\ &\quad \le 2P\{|X_{1,\lambda}(t_k)| > (\delta - \epsilon/2)\alpha_\lambda(t_{k+1})\}\\ &\quad \le 2P\{|X_{1,\lambda}(t_k)| > (\delta - \epsilon)\alpha_\lambda(t_k)\rho^{-1/p}\}.\end{aligned}$$

Thus $C_1(\lambda, \rho)$ will be finite if for some $\xi > 0$,

$$\sum_k P\{|X_{1,\lambda}(t_k)| > \xi\alpha_\lambda(t_k)\rho^{-1/p}\} < \infty. \tag{3.3}$$

By Lemma 2.1, for any $\epsilon > 0$ and $v > 0$ with $s = (1+\epsilon)L_2 t_k$,

$$\begin{aligned}&P\{|X_{1,\lambda}(t_k)| > 2^{-1}ve^v t_k b_\lambda(t_k)K(b_\lambda(t_k)) + v^{-1}(1+\epsilon)b_\lambda(t_k)L_2 t_k\}\\ &\quad \le \ 2exp(-(1+\epsilon)L_2 t_k).\end{aligned} \tag{3.4}$$

Now

$$\begin{aligned}&2^{-1}ve^v t_k b_\lambda(t_k)K(b_\lambda(t_k)) + v^{-1}(1+\epsilon)b_\lambda(t_k)L_2 t_k\\ &\quad \le \alpha_\lambda(t_k)(2^{-1}ve^v\lambda^{-1} + v^{-1}(1+\epsilon))\\ &\quad = \alpha_\lambda(t_k)\lambda^{-1/2}(2^{-1}e^{\lambda^{1/2}} + 1 + \epsilon)\end{aligned} \tag{3.5}$$

by setting $v = \lambda^{1/2}$. Since ϵ is arbitrary, (3.3) will follow from (3.4) and (3.5) if $\xi\rho^{-1/p} > \lambda^{-1/2}(2^{-1}e^{\lambda^{1/2}} + 1)$.
Thus $C_1(\lambda, \rho) \leq \rho^{1/p}\lambda^{-1/2}(2^{-1}e^{\lambda^{1/2}} + 1)$.

Proof of the part (a) of (1) in Theorem.

Recalling the definition of $\eta_2(\lambda)$ in (2.14), we see that for any $\lambda > 0$ we can find $\lambda_0 \in (0, \lambda]$ such that $\eta_2(\lambda_0) < (1+\theta)^{-1}$. Fix such a λ_0. Then for $\rho > 1$ and $t \in I(k)$,

$$X(t) - tM\{b_\lambda(t)\} = X_{1,\lambda}(t) + X_{2,\lambda}(t) + t\int xI\{b_\lambda(t) < |x| \leq b_\lambda(t_k)\}\, d\nu(x).$$

Setting $\eta_3(\lambda, \lambda_0) = [(1+\theta)^{-1} - \eta_2(\lambda_0)](\lambda_0/\lambda)^{2/p}\lambda_0^{-1}$,
we have by (2.17) and (3.1),

$$\begin{aligned}
&\limsup_{t\to 0} X_{1,\lambda}(t)/(T(t)L_2 t)^{1/2} \\
&\qquad \leq \limsup_{t\to 0} |X_{1,\lambda}(t)|/(\eta_3(\lambda,\lambda_0)\beta_\lambda(t)L_2 t)^{1/2} \\
&\qquad = C_1(\lambda,\rho)\eta_3(\lambda,\lambda_0)^{-1/2} < \infty.
\end{aligned} \tag{3.6}$$

Now (2.6) and (2.17) imply that

$$\begin{aligned}
&\limsup_{t\to 0} |t\int xI\{b_\lambda(t) < |x| \leq b_\lambda(t_k)\}\, d\nu(x)|/(T(t)L_2(t))^{1/2} \qquad (3.7)\\
&\qquad \leq \limsup_{t\to 0} t_k \int |x| I\{b_\lambda(t_{k+1}) < |x| \leq b_\lambda(t_k)\}\, d\nu(x)/\eta_3(\lambda,\lambda_0)^{1/2}\alpha_\lambda(t) \\
&\qquad \leq \eta_3(\lambda,\lambda_0)^{-1/2}\lambda^{-1}\rho^{1+1/p} < \infty.
\end{aligned}$$

Next by the Cauchy-Schwarz inequality,

$$\begin{aligned}
|X_{2,\lambda}(t)| &\leq \textstyle\sum_{s\leq t} |X(s) - X(s^-)| I\{|X(s) - X(s^-)| > b_\lambda(t_k)\} \\
&\leq [\textstyle\sum_{s\leq t}(X(s) - X(s^-))^2 I\{|X(s) - X(s^-)| > b_\lambda(t)\}]^{1/2} J_\lambda^{1/2}(t) \qquad (3.8)\\
&\leq (T(t)J_\lambda(t))^{1/2},
\end{aligned}$$

where $J_\lambda(t)$ is as in Lemma 2.5. Thus by Lemma 2.5,

$$\limsup_{t\to 0} |X_{2,\lambda}(t)|/(T(t)L_2 t)^{1/2} \leq \eta_1(\lambda)^{1/2}. \tag{3.9}$$

Combining (3.6), (3.7), (3.9) and zero-one law, the part (a) is proved.

4. LOWER BOUND

In this section, we prove the part (b) of (1) in Theorem.
To do this, we need to develop a suitable probability estimate. We first prove some lemmas, which lead to the necessary probability estimate. Throughout this section we will use the same notation as in Section III.

Lemma 4.1. Suppose that (2.1) holds and $\eta_2(\lambda)$ is as in (2.14) for $\lambda > 0$, and let

$$\xi > 1 + (1+\theta)\eta_2(\lambda). \tag{4.1}$$

Let $r \geq t$ be such that $r \sim t$ as $r,\ t \to 0$. Then for all t sufficiently small

$$P\{T_\lambda(t:r) > \xi ET_\lambda(t:r)\} \leq 2exp(-2L_2 t). \tag{4.2}$$

Proof. Applying Lemma 2.1 with $a = b_\lambda^2(r)$, $s = 2L_2 t$ and $v > 0$, we obtain

$$P\{|T_\lambda(t:r) - ET_\lambda(t:r)| > 2^{-1} v e^v t b_\lambda^2(r) K_\mu(b_\lambda^2(r)) + 2v^{-1} b_\lambda^2(r) L_2 t\}$$
$$\leq 2exp(-2L_2 t). \tag{4.3}$$

Noting that $K_\mu(x^2) \leq K(x)$, we obtain

$$K_\mu(b_\lambda^2(r)) \leq Q(b_\lambda(r)) = L_2 r/\lambda r. \tag{4.4}$$

By setting $v = \lambda^{1/2}$ and combining (4.3) and (4.4), we have

$$P\{|T_\lambda(t:r) - ET_\lambda(t:r)| > \lambda^{-1} b_\lambda^2(r) L_2 r (2^{-1}\lambda^{1/2} e^{\lambda^{1/2}} + 2\lambda^{1/2} L_2 t/L_2 r)\}$$
$$\leq 2exp(-2L_2 t). \tag{4.5}$$

Next notice that for all t sufficiently small,

$$\begin{aligned} ET_\lambda(t:r) &= t\int_{|x|\leq b_\lambda(r)} x^2\, d\nu(x) \\ &\geq t b_\lambda^2(r) Q(b_\lambda(r))/(1+\theta) \\ &= t b_\lambda^2(r) L_2 r/(\lambda r(1+\theta)). \end{aligned} \tag{4.6}$$

Now

$$
\begin{aligned}
P\{T_\lambda(t:r) > \xi ET_\lambda(t:r)\} &= P\{T_\lambda(t:r) - ET_\lambda(t:r) > (\xi-1)ET_\lambda(t:r)\} \\
&\le P\{|T_\lambda(t:r) - ET_\lambda(t:r)| > (\xi-1)ET_\lambda(t:r)\}
\end{aligned}
$$

and hence by (4.5)

$$P\{T_\lambda(t:r) > \xi ET_\lambda(t:r)\} \le 2exp(-2L_2 t) \tag{4.7}$$

provided that

$$(\xi-1)ET_\lambda(t:r) \ge \lambda^{-1} b_\lambda^2(r) L_2 r (2^{-1}\lambda^{1/2} e^{\lambda^{1/2}} + 2\lambda^{1/2} L_2 t / L_2 r). \tag{4.8}$$

If $\lambda > 0$ is fixed and the strict inequality in (4.1) holds, then $r \ge t$ and $r \sim t$ as $r,\ t \to 0$ together with (4.6) implies (4.8) for small t since $\eta_2(\lambda) = 2^{-1}\lambda^{1/2} e^{\lambda^{1/2}} + 2\lambda^{1/2}$.
Thus the lemma is proved.

Lemma 4.2. For r sufficiently small and $t \le r$,

$$P\{\sup_{s \le t} |X_{2,\lambda}(s:r)| = 0\} \ge exp(-\lambda^{-1} L_2 t). \tag{4.9}$$

Proof.

$$
\begin{aligned}
P\{\sup_{s \le t} |X_{2,\lambda}(s:r)| = 0\} &= exp[-tG(b_\lambda(r))] \\
&\ge exp[-tQ(b_\lambda(t))] \\
&= exp[-\lambda^{-1} L_2 t].
\end{aligned}
$$

Lemma 4.3. Suppose that (2.1) holds and let $r \ge t,\ r \sim t$ as $r,\ t \to 0$. If $\lambda > 0,\ \epsilon > 0$, and t is sufficiently small, there exist $\xi_1,\ \xi_2 \in (0,1)$ such that

$$P\{X_{1,\lambda}(t:r) > \delta[t b_\lambda^2(r) K(b_\lambda(r)) L_2 t]^{1/2}\} \ge exp(-\delta(1+\epsilon) L_2 t) \tag{4.10}$$

provided that

$$\xi_2 \lambda^{-1} \le \delta \le \xi_1 (tK(b_\lambda(r))/L_2 t)^{1/2}.$$

Proof. First we note that by assumption $tK(b_\lambda(r)) \to \infty$ as $t \to 0$. Thus the result follows immediately from a simple application of Lemma 2.2 since

$$\delta[tb_\lambda^2(r)K(b_\lambda(r))L_2t]^{1/2} \le \xi_1 tb_\lambda(r)K(b_\lambda(r))$$

and for t sufficiently small

$$\delta(1+\epsilon)L_2t \ge \xi_2 tQ(b_\lambda(r)) \ge \xi_2 tK(b_\lambda(r)).$$

Lemma 4.4. Suppose that (2.1) holds, and let $r \ge t$, $r \sim t$ as $r,\ t \to 0$. Then there exist $\xi_1,\ \xi_2 \in (0,1)$ such that if $\epsilon > 0$, $\lambda > \xi_2^2\xi_1^{-2}(1+\theta)$, $\xi > 1+(1+\theta)\eta_2(\lambda)$
and $\xi_2\lambda^{-1}\xi^{-1/2} < \delta < \xi_1\xi^{-1/2}(1+\theta)^{-1/2}\lambda^{-1/2}$

then for all t sufficiently small,

$$\begin{aligned} &P\{X(t) - tM(b_\lambda(r)) > \delta(T(t)L_2t)^{1/2}\} \\ &\quad \ge exp(-\lambda^{-1}L_2t)[exp(-\delta\xi^{1/2}(1+\epsilon)L_2t) - 2exp(-2L_2t)]. \end{aligned} \tag{4.11}$$

Proof. First we note that by Lemma 4.2,

$$\begin{aligned} &P\{X(t) - tM(b_\lambda(r)) > \delta(T(t)L_2t)^{1/2}\} \\ &\quad = P\{X_{1,\lambda}(t:r) + X_{2,\lambda}(t:r) > \delta(T(t)L_2t)^{1/2}\} \\ &\quad \ge P\{X_{1,\lambda}(t:r) > \delta(T(t)L_2t)^{1/2},\ X_{2,\lambda}(s,r) = 0 \text{ for all } s \le t\} \\ &\quad \ge P\{X_{1,\lambda}(t:r) > \delta(T_\lambda(t:r)L_2t)^{1/2}\}exp(-\lambda^{-1}L_2t). \end{aligned} \tag{4.12}$$

Furthermore,

$$\begin{aligned} &P\{X_{1,\lambda}(t:r) > \delta(T_\lambda(t:r)L_2t)^{1/2}\} \\ &\quad \ge P\{X_{1,\lambda}(t:r) > \delta(T_\lambda(t:r)L_2t)^{1/2},\ T_\lambda(t:r) \le \xi ET_\lambda(t:r)\} \\ &\quad \ge P\{X_{1,\lambda}(t:r) > \delta(\xi ET_\lambda(t:r)L_2t)^{1/2}\} - P\{T_\lambda(t:r) > \xi ET_\lambda(t:r)\}. \end{aligned} \tag{4.13}$$

Next observe that

$$ET_\lambda(t:r) = tb_\lambda^2(r)K(b_\lambda(r)), \tag{4.14}$$

and as $r,\ t \to 0$ with $r \sim t$

$$\xi_2\lambda^{-1} < \delta\xi^{1/2} < \xi_1(1+\theta)^{-1/2}\lambda^{-1/2}$$

$$\begin{aligned} &\sim \xi_1[tQ(b_\lambda(r))/(1+\theta)L_2t]^{1/2} \\ &\leq \xi_1[tK(b_\lambda(r))/L_2t]^{1/2}. \end{aligned}$$

Lemma 4.3 and (4.14) then imply that for any $\epsilon > 0$ if r, t are sufficiently small,

$$P\{X_{1,\lambda}(t:r) > \delta\xi^{1/2}(ET_\lambda(t:r)L_2t)^{1/2}\} \geq exp(-\delta\xi^{1/2}(1+\epsilon)L_2t). \quad (4.15)$$

Combining (4.12), (4.13) and (4.15) along with Lemma 4.1, we obtain that for t sufficiently small,

$$\begin{aligned} &P\{X(t) - tM(b_\lambda(r)) > \delta(T(t)L_2t)^{1/2}\} \\ &\geq exp(-\lambda^{-1}L_2t)[exp(-\delta\xi^{1/2}(1+\epsilon)L_2t - 2exp(-2L_2t))] \end{aligned}$$

provided that $\xi_2\lambda^{-1} < \delta\xi^{1/2} < \xi_1(1+\theta)^{-1/2}\lambda^{-1/2}$.

Thus the proof is completed.

Lemma 4.5. Suppose that (2.1) holds. Then for all λ sufficiently large, there exists a $\delta = \delta(\lambda) > 0$ such that for $r \geq t$, $r \sim t$ and t sufficiently small,

$$P\{X(t) - tM(b_\lambda(r)) > \delta(T(t)L_2t)^{1/2}\} \geq exp(-2^{-1}L_2t). \quad (4.16)$$

Proof. Set $\epsilon = 1$ in Lemma 4.4 and let

$$\lambda = 8 \vee (6\xi_2^2\xi_1^{-2}(1+\theta)) \vee (64\xi_1^2(1+\theta)^{-1}),$$

where ξ_1, ξ_2 are given as in Lemma 4.4.
Then for $\delta = 2^{-1}\xi_1\xi^{-1/2}(1+\theta)^{-1/2}\lambda^{-1/2}$, (4.16) follows from (4.11) since $1/\lambda \leq 1/8$ and

$$\delta\xi^{1/2}(1+\epsilon) = \xi_1(1+\theta)^{-1/2}\lambda^{-1/2} \leq 1/8.$$

The extra term $2exp(-2L_2t)$ in (4.11) is small enough to be absorbed into the right-hand side of (4.16).

Lemma 4.6. If $t_k = exp(-k^q)$ with $q > 1$, then

$$\lim_k T(t_{k+1})/T(t_k) = 0 \ a.s.. \tag{4.17}$$

Proof. It suffices to show that for every $\epsilon > 0$,

$$\sum_k P\{T(t_{k+1}) > \epsilon T(t_k)\} < \infty. \tag{4.18}$$

Fix $\epsilon > 0$ and without loss of generality we assume $1/\epsilon$ is an integer. Let $n_k = [t_k/t_{k+1}]$ and $W_j = T(jt_{k+1}) - T((j-1)t_{k+1})$ for $j = 1, 2, \ldots, n_k$. Then $T(t_k) \geq W_1 + W_2 + \ldots + W_{n_k}$ and $\lim_k n_k = \infty$.
Now by Lemma 5.7 of Griffin and Kuelbs(1989),

$$\begin{aligned} P\{T(t_{k+1}) > \epsilon T(t_k)\} &\leq P\{W_1 > \epsilon(W_1 + \ldots + W_{n_k})\} \\ &\leq P\{\sum_{j=1}^{n_k} I\{W_j \geq W_1\} < 1/\epsilon\} \\ &\leq (\epsilon n_k)^{-1}. \end{aligned}$$

Since $n_k \sim exp((k+1)^q - k^q) \geq exp(qk^{q-1})$ for large k with $q > 1$, (4.18) follows immediately. Thus the lemma is proved.

Proof of the part (b) of (1) in Theorem.

Let $t_k = exp(-k^q)$ with $1 < q < 2$ and write

$$X(t_k) - t_k M(b_\lambda(t_k)) \tag{4.19}$$

$$= (X(t_k) - X(t_{k+1})) - (t_k - t_{k+1})M(b_\lambda(t_k)) + X(t_{k+1})$$

$$-t_{k+1}M(b_\lambda(t_{k+1})) - t_{k+1}\int xI\{b_\lambda(t_{k+1}) < |x| \leq b_\lambda(t_k)\}\, d\nu(x).$$

By the part (a) of (1) in Theorem and Lemma 4.6, for all $\lambda > 0$

$$\limsup_k |X(t_{k+1}) - t_{k+1}M(b_\lambda(t_{k+1}))|/(T(t_k)L_2 t_k)^{1/2} = 0 \ \ a.s.. \tag{4.20}$$

Next by (2.17), for each $\lambda > 0$, there is a $\xi = \xi(\lambda) > 0$ such that

$$\liminf_{t\to 0} T(t)/\beta_\lambda(t) \geq \xi.$$

Hence by the Cauchy-Schwarz inequality,

$$\begin{aligned}\limsup_k t_{k+1}\Big|\int xI\{b_\lambda(t_{k+1}) < |x| \leq b_\lambda(t_k)\}\, d\nu(x)\Big|/(T(t_k)L_2t_k)^{1/2} \qquad (4.21)\\ \leq \limsup_k t_{k+1}\Big[\int x^2 I\{|x| \leq b_\lambda(t_k)\}\, d\nu(x)\Big]^{1/2} G(b_\lambda(t_{k+1}))^{1/2}/(\xi\beta_\lambda(t_k)L_2t_k)^{1/2}\\ \leq \limsup_k (t_{k+1}L_2t_{k+1}/\lambda^2\xi t_k L_2 t_k)^{1/2} = 0.\end{aligned}$$

Applying now Lemma 4.5 with $t = t_k - t_{k+1}$ and $r = t_k$, it follows that for $\lambda > 0$ sufficiently large there is a $\delta = \delta(\lambda) > 0$ such that for k sufficiently large

$$\begin{aligned}P\{X(t_k)-X(t_{k+1})-(t_k-t_{k+1})M(b_\lambda(t_k)) > \delta[(T(t_k)-T(t_{k+1}))L_2(t_k-t_{k+1})]^{1/2}\}\\ \geq exp(-2^{-1}L_2(t_k - t_{k+1})) \sim k^{-q/2}. \qquad (4.22)\end{aligned}$$

Finally by Lemma 4.6,

$$(T(t_k) - T(t_{k+1}))L_2(t_k - t_{k+1}) \sim T(t_k)L_2t_k.$$

Thus, since the events in (4.22) are independent, the Borel-Cantelli lemma and (4.19) $\sim$ (4.21) imply

$$\limsup_{t\to 0}[X(t) - tM(b_\lambda(t))]/(T(t)L_2t)^{1/2} \geq \delta > 0 \ a.s.$$

which completes the proof.

REFERENCE

(1) Ahn, W.C. and Wee,I.S.(1985). *The class of limit Laws for Levy processes*, Journal of the Korean Mathematical Society. Vol.22, 199-210.

(2) Bingham,N.H.(1986). *Variants on the law of the iterated logarithm*, Bulletin of the London Mathematical Society. Vol. 18, 433-467.

(3) Blumenthal,R.M. and Getoor,R.K.(1961). *Sample functions of stochastic processes with stationary independent increments.* Journal of Mathematical Mechanics, Vol. 10, 493-561.

(4) Feller,W.(1968). *An extension of the law of the iterated logarithm to variables without variance.* Journal of Mathematical Mechanics, Vol. 18, 343-355.

(5) Fristedt,B.E.(1974). *Sample functions of stochastic processes with stationary independent increments.* Advances in Probability and Related Topics, Vol.3, Marcel Dekker, New York, 241-396.

(6) Gihman,I.I. and Skorokhod,A.V.(1975). *The theory of stochastic processes II*, Sprigner-Verlag, New York.

(7) Griffin,Ph.S.(1985). *Laws of the iterated logarithm for symmetric stable processes.* Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete, Vol. 68, 271-285.

(8) Griffin,Ph.S. and Kuelbs,J.D.(1989). *Self-normalized laws of the iterated logarithm.* Annals of Probability, Vol.17, 1571-1601.

(9) Kim,Y.K. and Wee, I.S.(1988). *General laws of the iterated logarithm for Levy processes.* Journal of the Korean Statistical Society, Vol.17, 30-40.

(10) Kim, Y.K. and Wee, I.S.(1991). *Upper functions for Levy processes having only negative jumps.* Stochastic Analysis and Applications, Vol. 9, 301-310.

(11) Pruitt,W.E.(1981). *General one-sided laws of the iterated logarithm.* Annals of Probability, Vol.9, 1-48.

(12) Wee,I.S.(1988). *Lower functions for processes with stationary independent increments.* Probability Theory and Related Fields, Vol.77, 551-566.