Journal of the Korean
OR/ MS Society
Vol. 18, No. 3, December 1993



# System Size and Service Size Distributions of a Batch Service Queue†

Soon Seok Lee*, Ho Woo Lee**, Seung Hyun Yoon** and R. Nadarajan***

**Abstract**

We derive the arbitrary time point system size distribution of $M/G^B/1$ queue in which late arrivals are not allowed to join the on-going service. The distribution is given by $P(z)=P_q(z)\ S^*(\lambda-\lambda z)$ where $P_q(z)$is the probability generating function of the queue size and $S^*(\theta)$ is the Laplace-Stieltjes transform of the service time distribution function. We also derive the distribution of the service size at an arbitrary point of time.

## 1. Introduction

Batch service queues were studied by many researchers. They have been applied to traffic systems, transportation systems, and production & manufacturing systems. The first study on batch service queues was due to Bailey [1]. He obtained the transform solution to the fixed size batch service queue with Poisson arrivals. Other studies followed on general bulk service (Neuts [8]), batch arrival, batch service (Miller [7]), and random size batch service (Jaiswal [4]). For more comprehensive discussion in this or related topics, readers are urged to see Chaudhry and Templeton [2]. Lee and Lee [6] studied the decomposition of batch service queues.

Most of the studies dealt with queue size distribution instead of system size distribution because "it is more convenient to derive the queue size distribution" (Chaudhry and Templeton [2]). The difficulty in modeling the system size instead of queue size occurs in a lot of batch

† This is a part of the research supported by KOSEF #921-0900-003-2.

* ATM Lab, ETRI

** Department of Industrial Engineering, Sung Kyun Kwan University

*** Department of Math. & Computer Application, PSG College of Technology, INDIA

service queueing systems. We in this paper suggests a modelling technique that makes it easy to derive the system size distribution as well as the service size distribution. This is accomplished by keeping track of the "service size" all the time.

In this paper we consider a batch service queueing system with following specifications (this queueing system is denoted as $M/G^B/1$ queue in queueing literature),

(1) The arrival process is Poisson with rate $\lambda$.

(2) The service time follows a general distribution.

(3) As soon as a service is finished, the server takes into service maximum of $B$ customers ($B$ is called the "service capacity").

(4) If less then $B$ customers are present at a service completion epoch, all the existing customers are taken into service.

(5) Those customers who arrive while a group of customers are being served cannot join the service even if there is a space available.

## 2. System Size Distribution

In this section, we derive the system size distribution of the above queueing. We model the system by using the elapsed service time as the supplementary variable.

Let $\{N_q(t),\ t\geq 0\}$ be the queue size process. Since $N_q(t)$ is not Markovian, it is impossible to set up the system equations for $N_q(t)$. It is customary to add $\widetilde{S}(t)$, the elapsed service time of the customer in service at time $t$, as the supplementary variable. Then we have the bivariate process $\{N_q(t),\ \widetilde{S}(t)\}$. As far as the queue size is concerned, the bivariate process contains all the information to set up the system equations. But if we are interested in deriving the system size distribution, we need to add one more variable $U(t)$ which is the number of customers currently in service at time $t$. This forms a trivariate process $\{N_q(t),\ \widetilde{S}(t),\ U(t)\}$. It is easy to show that the trivariate process is Markovian.

Let us define the following notations and probabilities,

$\lambda$ – *arrival rate*,

$\mu$ – *service rate*,

$B$ – *service capacity*,

$\rho$ – *traffic intensity* $(=\frac{\lambda}{\mu B})$,

$S$ − *service size random variable,*
$S(x), s(x)$ − *service time distribution function and density,*
$S^*(\theta)$ − *Laplace-Stieltjes transfom (LST) of $S(x)$,*
$\tilde{S}(t)$ − *elapsed service time of the customer in service at time $t$,*
$U(t)$ − *service size at time $t$, ($U(t)=0$ if server is idle)*
$N(t)$ − *system size at time $t$,*
$N_q(t)$ − *queue size at time $t$.*

$$Q_{n,m}(x,t)\Delta t = Pr[N_q(t)=n,\ U(t)=m,\ x\le \tilde{S}(t)\le x+\Delta t],\ (n\ge 0),\ m=1,2,\cdots,B,$$
$$Q_{00}(t) = Pr[N(t)=0],$$
$$\eta(x)dx = Pr[\text{a service ends in } (x,x+dx) \text{ given that it has not finished in time } x].$$

It is not difficult to derive following infinitesimal system equations.

$$Q_{00}(t+dt) = Q_{00}(t)(1-\lambda dt) + \sum_{m=1}^{B}\int_0^\infty Q_{0,m}(x,t)\eta(x)dx\ dt(1-\lambda dt) + O(dt),$$
$$Q_{0,m}(x+dt,t+dt) = Q_{0,m}(x,t)(1-\lambda dt)(1-\eta(x)dt) + O(dt),\quad m=1,2,\cdots,B$$
$$Q_{n,m}(x+dt,t+dt) = Q_{n,m}(x,t)(1-\lambda dt)(1-\eta(x)dt) + Q_{n-1,m}(x,t)\lambda dt + O(dt),\quad n\ge 1,\quad m=1,2,\cdots,B.$$

The first equation states that there are no customers in the system at time $t+dt$ iff there are no arrival and no service completion in $(t,t+dt)$. Similar interpretations could be given to the remaining equations. From above equations we can easily obtain following steady-state system equations.

$$0 = -\lambda Q_{00} + \sum_{m=1}^{B}\int_0^\infty Q_{0,m}(x)\eta(x)dx, \tag{1}$$
$$(d/dx)Q_{0,m}(x) = -[\lambda+\eta(x)]Q_{0,m}(x),\quad m=1,2,\cdots,B, \tag{2}$$
$$(d/dx)Q_{n,m}(x) = -[\lambda+\eta(x)]Q_{n,m}(x) + \lambda Q_{n-1,m}(x)\quad (n\ge 1),\quad m=1,2,\cdots,B. \tag{3}$$

The steady-state boundary conditions are,

$$Q_{0,1}(0) = \lambda Q_{00} + \sum_{k=1}^{B}\int_0^\infty Q_{1,k}(x)\eta(x)dx, \tag{4}$$
$$Q_{0,m}(0) = \sum_{k=1}^{B}\int_0^\infty Q_{m,k}(x)\eta(x)dx,\quad m=2,3,\cdots,B-1 \tag{5}$$
$$Q_{n,B}(0) = \sum_{k=1}^{B}\int_0^\infty Q_{n+B,k}(x)\eta(x)dx,\quad (n\ge 0). \tag{6}$$

Let us define following probability generating functions ($PGF$).

$$\Pi_m(z_1,x) = \sum_{n=0}^{\infty} Q_{n,m}(x)z_1^n, \tag{7}$$

$$\Pi(z_1,z_2,x) = \sum_{m=1}^{B} \Pi_m(z_1,x)z_2^m, \tag{8}$$

$$\Pi(z_1,z_2) = \int_0^\infty \Pi(z_1,z_2,x)dx, \tag{9}$$

$$P(z_1,z_2) = Q_{00} + \Pi(z_1,z_2), \tag{10}$$

$$P(z) = P(z,z) = Q_{00} + \Pi(z,z), \tag{11}$$

$$P_q(z) = Q_{00} + \Pi(z,1). \tag{12}$$

Notice that $P(z)$ and $P_q(z)$ are the *PGF*s of the system size and the queue size at arbitrary time points respectively. From equations (2) and (3), we have

$$\frac{\partial}{\partial x} \Pi_m(z_1,x) = [\lambda z_1 - \lambda - \eta(x)]\Pi_m(z_1,x), \quad m=1,2,\cdots,B. \tag{13}$$

The solution of equation (13) is given by,

$$\Pi_m(z_1,x) = C \cdot [1-S(x)]e^{-\lambda(1-z_1)x}.$$

*By letting* $x$ go to 0, we have

$$\Pi_m(z_1,x) = \Pi_m(z_1,0)\ [1-S(x)]e^{-\lambda(1-z_1)x}. \tag{14}$$

From boundary conditions (4), (5) and (6), we have

$$\Pi_1(z_1,0) = \lambda Q_{00} + \sum_{k=1}^{B} \int_0^\infty Q_{1,k}(x)\eta(x)dx, \tag{15}$$

$$\Pi_j(z_1,0) = \sum_{k=1}^{B} \int_0^\infty Q_{j,k}(x)\eta(x)dx, \quad j=2,3,\cdots,B-1 \tag{16}$$

$$\begin{aligned}\Pi_B(z_1,0) &= \sum_{n=0}^{\infty} z_1^n \sum_{k=1}^{B} \int_0^\infty Q_{n+B,k}(x)\eta(x)dx \\ &= (1/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=B}^{\infty} z_1^n Q_{n,k}(x)\eta(x)dx \\ &= (1/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{\infty} z_1^n Q_{n,k}(x)\eta(x)dx - (1/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{B-1} z_1^n Q_{n,k}(x)\eta(x)dx.\end{aligned} \tag{17}$$

Now,

$$\begin{aligned}\Pi(z_1,z_2,x) &= \sum_{m=1}^{B} z_2^m \Pi_m(z_1,x) \\ &= \sum_{m=1}^{B} z_2^m \Pi_m(z_1,0)[1-S(x)]e^{-\lambda(1-z_1)x} \\ &= \Pi(z_1,z_2,0)[1-S(x)]e^{-\lambda(1-z_1)x}.\end{aligned} \tag{18}$$

From equations (15),(16) and (17), we have

$$
\begin{aligned}
\Pi(z_1,z_2,0) &= \sum_{m=1}^{B} z_2^m\ \Pi_m\ (z_1,0) \\
&= \lambda z_2 Q_{00} + z_2 \sum_{k=1}^{B} \int_0^\infty Q_{1,k}(x)\eta(x)dx + \sum_{j=2}^{B-1} z_2^j \sum_{k=1}^{B} \int_0^\infty Q_{j,k}(x)\eta(x)dx \\
&\quad + (z_2^B/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{\infty} z_1^n Q_{n,k}(x)\eta(x)dx - (z_2^B/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{B-1} z_1^n Q_{n,k}(x)\eta(x)dx \\
&= \lambda(z_2-1)Q_{00} + (z_2/z_1)^B \int_0^\infty \Pi(z_1,1,x)\eta(x)dx \\
&\quad + \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{B-1} [z_2^n - z_1^n(z_2/z_1)^B]\ Q_{n,k}(x)\eta(x)dx \\
&= \lambda(z_2-1)Q_{00} + \sum_{k=1}^{B} \sum_{n=0}^{B-1} [z_2^n - z_1^n(z_2/z_1)^B] \int_0^\infty Q_{n,k}(x)\eta(x)dx \\
&\quad + (z_2/z_1)^B \int_0^\infty \Pi(z_1,1,0)e^{\lambda(1-z_1)x} \cdot [1-S(x)]\eta(x)dx \\
&= \lambda(z_2-1)Q_{00} + \sum_{k=1}^{B} \sum_{n=0}^{B-1} [z_2^n - z_1^n(z_2/z_1)^B] \int_0^\infty Q_{n,k}(x)\eta(x)dx \\
&\quad + (z_2/z_1)^B\ S^*(\lambda-\lambda z_1)\ \Pi(z_1,1,0).
\end{aligned}
\tag{19}
$$

Thus

$$
\begin{aligned}
\Pi(z_1,1,0) &= \sum_{m=1}^{B} \Pi_m(z_1,0) \\
&= \lambda Q_{00} + \sum_{n=1}^{B-1} \sum_{k=1}^{B} \int_0^\infty Q_{n,k}(x)\eta(x)dx \\
&\quad + (1/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{\infty} z_1^n Q_{n,k}(x)\eta(x)dx - (1/z_1^B) \sum_{k=1}^{B} \int_0^\infty \sum_{n=0}^{B-1} z_1^n Q_{n,k}(x)\eta(x)dx \\
&= \lambda Q_{00} + \sum_{k=1}^{B} \sum_{n=0}^{B-1} \int_0^\infty (1-z_1^{n-B})\ Q_{n,k}(x)\eta(x)dx \\
&\quad - \sum_{k=1}^{B} \int_0^\infty Q_{0,k}(x)\eta(x)dx + \frac{S^*(\lambda-\lambda z_1)\Pi(z_1,1,0)}{z_1^B}.
\end{aligned}
$$

Solving for $\Pi(z_1,1,0)$ yields

$$
\Pi(z_1,1,0) = \frac{\sum_{k=1}^{B} \sum_{n=0}^{B-1} \int_0^\infty (z_1^B - z_1^n)Q_{n,k}(x)\eta(x)dx}{z_1^B - S^*(\lambda-\lambda z_1)}.
\tag{20}
$$

Therefore, we have

$$
\begin{aligned}
\Pi(z_1,z_2,0) &= \lambda(z_2-1)Q_{00} + \sum_{k=1}^{B} \sum_{n=0}^{B-1} \int_0^\infty [z_2^n - z_1^n(z_2/z_1)^B]\ Q_{n,k}(x)\eta(x)dx \\
&\quad + (z_2/z_1)^B S^*(\lambda-\lambda z_1) \cdot \frac{\sum_{k=1}^{B} \sum_{n=0}^{B-1} \int_0^\infty (z_1^B - z_1^n)Q_{n,k}(x)\eta(x)dx}{z_1^B - S^*(\lambda-\lambda z_1)}.
\end{aligned}
\tag{21}
$$

From definition (9) and equation (18), we have

$$\Pi(z_1,z_2) = \Pi(z_1,z_2,0)\cdot\frac{1-S^*(\lambda-\lambda z_1)}{\lambda(1-z_1)}. \tag{22}$$

Letting $z_2 = z_1 = z$ in equation (22) yields

$$\Pi(z,z) = \frac{1-S^*(\lambda-\lambda z)}{\lambda(1-z)}\times$$
$$\{\ \lambda(z-1)Q_{00} + \frac{S^*(\lambda-\lambda z)\sum_{k=1}^{B}\sum_{n=0}^{B-1}\int_0^\infty (z^B-z^n)Q_{n,k}(x)\eta(x)dx}{z^B-S^*(\lambda-\lambda z)}\ \}.$$

From equation (11), we have the system size distribution at an arbitrary time point,

$$\begin{aligned} P(z) &= Q_{00} + \Pi(z,z) \\ &= Q_{00} + [S^*(\lambda-\lambda z)-1]Q_{00} \\ &\quad + S^*(\lambda-\lambda z)\cdot\frac{[1-S^*(\lambda-\lambda z)]\sum_{k=1}^{B}\sum_{n=0}^{B-1}\int_0^\infty (z^B-z^n)Q_{n,k}(x)dx}{\lambda(1-z)[z^B-S^*(\lambda-\lambda z)]} \\ &= S^*(\lambda-\lambda z)\cdot\{\ Q_{00} + \frac{[1-S^*(\lambda-\lambda z)]\sum_{k=1}^{B}\sum_{n=0}^{B-1}\int_0^\infty (z^B-z^n)Q_{n,k}(x)dx}{\lambda(1-z)[z^B-S^*(\lambda-\lambda z)]}\ \}. \end{aligned} \tag{23}$$

The term in the bracket of equation (23) is the queue size *PGF* at an arbitrary time point (see Chaudhry and Templeton [2 : page 183]. Thus we have

$$P(z) = S^*(\lambda-\lambda z)\cdot P_q(z). \tag{24}$$

# 3. Service Size Distribution

Let $U_n=\lim_{t\to\infty} Pr[U(t)=n]$, $n=0,1,2,\cdots,B$ be the probability that the service size is $n$ at an arbitrary time point. If the server is idle, the service size is zero. Let $U(z_2) = \sum_{n=0}^{B} u_n z_2^n$ be the service size *PGF*. Then we have

$$U(z_2) = Q_{00} + \Pi(1,z_2) \tag{25}$$

From eq. (22) and the L'hospital's rule, we have:

$$U(z_2) = Q_{00} + B\rho(z_2-1)Q_{00} - E(S)\sum_{n=0}^{B-1}\sum_{m=1}^{B}(z_2^B - z)\int_0^\infty Q_{n,m}(x)\eta(x)dx$$

$$+ z_2^B \cdot \frac{E(S)\sum_{n=0}^{B-1}\sum_{m=1}^{B}(B-n)\int_0^\infty Q_{n,m}(x)\eta(x)dx}{B(1-\rho)}$$

$$= Q_{00} + B\rho(z_2-1)Q_{00} - E(S)\sum_{n=0}^{B-1}\sum_{m=1}^{B}(z_2^B - z)\int_0^\infty Q_{n,m}(x)\eta(x)dx + (1-Q_{00})z_2^B, \quad (26)$$

where the last equality comes from Chaudhry & Templeton [2].

Recovering the coefficients of $z_2^n$, $n=0,1,\cdots,B$, we have

$$U_0 = Q_{00},$$

$$U_1 = B\rho\, Q_{00} + E(S)\sum_{m=1}^{B}\int_0^\infty Q_{1,m}(x)\eta(x)dx,$$

$$U_n = E(S)\sum_{m=1}^{B}\int_0^\infty Q_{n,m}(x)\eta(x)dx, \quad (n=2,3,\cdots,B-1).$$

$$U_B = 1-Q_{00}-E(S)\sum_{n=0}^{B-1}\sum_{m=1}^{B}\int_0^B Q_{n,m}(x)\eta(x)dx.$$

Let $P_n^+$ be the departure point system size distribution. Then from Chaudhry & Templeton [2],

$$Q_{00} = \frac{P_0^+}{B\rho+P_0^+},$$

$$\sum_{m=1}^{B}\int_0^\infty Q_{n,m}(x)\eta(x)dx = \frac{1-Q_{00}}{E(S)}\cdot P_n^+.$$

Thus we have the service size distributions,

$$U_0 = \frac{P_0^+}{B\rho+P_0^+},$$

$$U_1 = \frac{B\rho}{B\rho+P_0^+}\cdot(P_0^+ + P_1^+),$$

$$U_n = \frac{B\rho P_n^+}{B\rho+P_0^+}, \quad (n=2,3,\cdots,B-1),$$

$$U_B = \frac{B\rho\{1-\sum_{n=0}^{B-1}P_n^+\}}{B\rho+P_0^+}. \quad (27)$$

The departure point system size distributions $\{P_n^+,\ n=0,1,2,\cdots,B\}$ can be obtained by applying the Rouche's theorem to the departure point system size *PGF* (Chaudhry & Templeton [2]).

$$P^{+}(z) = \frac{S^{*}(\lambda-\lambda z)\sum_{j=0}^{B}(z^{B}-z^{j})\cdot P_{j}^{+}}{z^{B}-S^{*}(\lambda-\lambda z)}.$$

The distribution of the conditional service size given that the server is busy becomes,

$$\widetilde{U}_{n} = \frac{U_{n}}{1-Q_{00}}, \quad n=1,2,\cdots,B. \tag{28}$$

# Reference

[1] Bailey, N.T.J., "On Queueing Process with Bulk Service," *J.Roy. Stat. Soc.*, B16(1954), pp. 80-97.

[2] Chaudhry, M.L. and J.G.C. Templeton, *A First Course in Bulk Queues*, Wiley, New York, 1983.

[3] Downton, F., "Waiting Time in Bulk Service Queues," *J. Roy. Stat. Soc.*, B17(1955), pp.256-261.

[4] Jaiswal, N.K., "A Bulk Service Queueing Problem with Variable Capacity," *J. Roy. Stat. Soc.*, B26(1964), pp.143-148.

[5] Kambo, N.S. and M.L. Chaudhry, "A Single-server Bulk-service Queue with Varying Capacity and Erlang Input," *INFOR*, Vol.23, No.2(1985), pp.196-204.

[6] Lee, H.W. and S.S. Lee, "Decompositions in batch service queues," *Research Report*, Dept. of Industrial Engineering, Sung Kyun Kwan University, Su Won, KOREA, 1993, pp.440-746.

[7] Miller, R.G., "A Contribution to the Theory of Bulk Queues," *J. Roy. Stat. Soc.*, B21(1959), pp.320-337.

[8] Neuts, M.F., "A General Class of Bulk Queues with Poisson Input," *Ann. Math. Stat.*, Vol. 38(1967), pp.757-770.