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Job Sequencing Problem for Three-Machine Flow Shop with Fuzzy Processing Times

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Abstract

This paper presents possibilistic job scheduling based on the membership function as an alternative to probabilistic job scheduling and illustrates a methodology for solving job sequencing problem which the opinions of experts greatly disagree in each processing time. Triangular fuzzy numbers are used to represent the processing times of experts. Here, the comparison method is based on the dominance property. The criteria for dominance are presented. By the dominance criteria, for each job, a major TFN and a minor TFN are selected and a pessimistic sequence with major TFNs and an optimistic sequence with minor TFNs are computed. The three-machine flow shop problem is considered as an example to illustrate the approach.

1. Introduction

In general, the processing times are assumed to be known exactly by experts. However, in practical situations, this is seldom the case. Occasionally, a manager is challenged by job sequencing problems with which she/he have had no prior experience. Therefore, the processing times can only be estimated as within certain intervals. This processing time interval can be naturally represented by a fuzzy number. In flow ship scheduling, the job sequencing problem is to order the jobs through the machines in such a way as to optimize certain performance criteria. One of the problems in solving job sequencing problems by considering fuzzy numbers is comparing the fuzzy numbers to obtain a total order. Prade [12, 13], Dubois and Prade [3] and Dumitru and Luban [4] have applied the fuzzy set theory to job sequencing problems. These approaches used only some threshold values to comparison fuzzy numbers and are limited in scope. Serveral effective comparison methods (ranking methods) have been devel-

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oped recently [2, 8, 9]. By the use of these ranking methods, the job sequencing problems with fuzzy processing times can be solved effectively. Recently, McCahon and Lee [10, 11] have solved efficiently the job sequencing problems with fuzzy processing times using Lee-Li's [6] ranking method based on a generalized mean value and spread of the fuzzy numbers. Occasionally, the opinions of many experts are required to represent uncertain processing times in a large scale flow shop. In such cases, for each processing time, many experts are interviewed and they can express processing times as fuzzy numbers.

This paper shows a methodology for solving job sequencing problem which the opinions of experts greatly disagree in each processing time. Triangular fuzzy numbers(TFNs) are used to represent processing times of experts. The objective of this paper is to introduce a pessimistic sequence with major TFNs and an optimistic sequence with minor TFNs and illustrate the approach using an effective method. The three-machine flow shop problem for makespan is considered as an example to illustrate the approach. Branch & bound algorithm is then modified to accept fuzzy job processing times.

2. Fuzzy Processing Times

A fuzzy number can be represented using the concept of an interval of confidence as in Kaufmann and Gupta [6]. It should be emphasized that a fuzzy number is essentially a generalized interval of confidence. An interval of confidence is one way of reducing the uncertainty of using lower and upper bounds. Also, a fuzzy number is a subjective datum. It is a valuation, not a measure. Let us relate the concept of the interval of confidence to another called the level of presumption. Let us assume, for example, that a certain job is to be completed between two dates, say May 15 and May 31. This is an interval of confidence. On the other hand, let us assume that this same job is to be completed on May 22, possible date. The interval of confidence in the first case is [May 15, May 31] while in the second case it is [May 22, May 22]. If we wish, we may assign two levels of confidence to these two situations, 0 for [May 15, May 31] and 1 for [May 22, May 22]. These two levels of confidence are in fact levels of presumption. In general, we can represent them between 0 and 1.

A triangular fuzzy number(TFN) can be defined by a triplet (a_1, a_2, a_3). The TFN which is a special kind of fuzzy number can represent the estimated processing time naturally. For example, an expert may say that the processing time for job A is generally a_2 min. But, due to other factors which cannot be controlled, the processing time may be occasionally as slow as a_3

min or as fast as a_1 min. This result is naturall a TFN. The membership function of a TFN is defined as

$$\mu_{\tilde{A}}(X) = 0,\ x < a_1,$$

$$= \frac{x-a_1}{a_2-a_1},\ a_1 \le x \le a_2,$$

$$= \frac{a_3-x}{a_3-a_2},\ a_2 \le x \le a_3,$$

$$= 0,\ x > a_3,$$

where $\mu_{\tilde{A}}(X)$ is the degree of membership or membership function value of x in $\tilde{A}$.

3. A Ranking method for Solvi ıg job Sequencing Problem

In this paper, ranking method based on the dominance property [7] is used. By using this method, we can solve efficiently the job seqı encing problem which the opinions of experts greatly disagree in each processing time. Let u consider a sheaf G composed of TFNs $\tilde{A}_i$ i=1, 2,…,n. We define $\tilde{A}^*$ as a major TFN if this TFN dominates all the other Ai in the sheaf G. The criterion for dominance is one of the follow ng three in the order given below.

(1) The greatest associated ordinary number as

$$A^* = \left(\frac{a_1+2a_2+a_3}{4}\right)^*$$

(2) If (1) does not separate the two TFNs, those which have the best maximum presumption(the mode) will be chosen.

(3) If (1) and (2) do not separate the TFNs, the divergence(the distance between two end points) will be used as the third criterion.

On the contrary, we call TFN $\tilde{A}^*$ a minor TFN if the TFN is dominated by all others in the sheaf G using respectively the criteria (1), (2) and (3).

4. Three-Machine Flow Shop Problem with TFN Processing Times

We will assume that all jobs are available fc processing immediately. The optimal sequence is defined as the sequence which minimizes the makespan. The length of time required to complete all jobs is called the makespan. The three machine flow shop problem with the objective of minimizing makespan can be solved more efficiently by Ignall and Schrage's branch & bound algorithm [1, 5]. Also, a fuzzified branch & bound algorithm can solve the three-machine flow shop problem with TFN processing times naturally. For a given partial sequence σ, let $\tilde{LC}_1(\sigma)$, $\tilde{LC}_2(\sigma)$ and $\tilde{LC}_3(\sigma)$ be the completion time of the last job on machine 1, 2 and 3, respectively, among jobs in σ. The lower bounds of the makespans on machine 1, 2 and 3 respectively, are

$$\tilde{lb}_1 = \tilde{LC}_1(\sigma) + \sum_{i\in\delta} \tilde{P}_{1i} + \min_{i\in\delta} \{\tilde{P}_{2i} + \tilde{P}_{3i}\}$$

$$\tilde{lb}_2 = \tilde{LC}_2(\sigma) + \sum_{i\in\delta} \tilde{P}_{2i} + \min_{i\in b\delta} \{\tilde{P}_{3i}\}$$

$$\tilde{lb}_3 = \tilde{LC}_3(\sigma) + \sum_{i\in\delta} \tilde{P}_{3i}$$

where $\tilde{P}_{ji}$ is the fuzzy processing time of job i on machine j and δ denote the set of jobs that are not contained in the partial sequence σ.

Then, the lower bound on makespan is

$$\tilde{LB}(\sigma) = \max\{\tilde{lb}_1, \tilde{lb}_2, \tilde{lb}_3\}.$$

Fuzzy makespan can be expressed as

$$\tilde{m}_s = \tilde{max}\ \tilde{C}_{3i(k)},$$

where $\tilde{C}_{3i(k)}$ is the fuzzy completion time of the kth job i in order on machine 3. The operators of max and min compared by the dominance criteria correspond to major and minor respectively. The operator of $\tilde{max}$ is the fuzzy maximum of fuzzy numbers.

Fuzzy mean flow time can be expressed as

$$\tilde{m}_{FT} = \frac{1}{n} \sum_{k=1}^{n} \tilde{C}_{3i(k)}'$$

where $\tilde{C}_{3i(k)} = \sum_{j=1}^{3} (\tilde{q}_{j(k)} + \tilde{p}_{ji(k)})$.

The fuzzy waiting time for machine 3 is

$$\tilde{q}_{3(k)} = \tilde{C}_{3i(k-1)} - \tilde{C}_{2i(k)}.$$

We can express $P^v_{ij}=(a_v, b_v, c_v)$ as expert v's fuzzy processing time for job i on machine j.

5. Fuzzified Branch & Bound Algorithm

Ignall and Schrage's branch & bound algorithm is modified for the pessimistic and optimistic sequences of three-machine flow shop problem with multiple TFN processing times. Then, an optimal fuzzy makespan can be constructed by the following algorithm.

[Step 1] Set j=1, i=1.
[Step 2] Find a major and minor TFN.
[Setp 3] (a) If it is the pessimistic sequence, set major=$\tilde{P}_{ji}$.
(b) If it is the optimistic sequence, set minor=$\tilde{P}_{ji}$.
[Setp 4] If i=n, go to Step 5. Otherwise, return to Step 2 with i=i+1.
[Setp 5] If j=3, go to Step 6. Otherwise, return to Step 2 with j=j+1, i=1.
[Setp 6] Let $\tilde{P}_{ji}$=processing time of job i on machine j and r=number of jobs in σ.
[Setp 7] Set r=1.
[Setp 8] Calculate $\tilde{LB}(\sigma)$ for σ respectively.
[Setp 9] Find a partial sequence node with the least lower bound $\tilde{LB}(\sigma)$. If this is a partilal sequence node with least lower bound, go to Step 10. Otherwise, go to Step 10 with new partial sequence node with least lower bound $LB(\sigma)$.
[Step 10] If r=n, stop. Otherwise, go to Step 11.
[Step 11] Branch from this node to node with r=r+1 and return to Step 8.

6. Example

To illustrate the proposed approach, a four-job three-machine flow shop problem is presented as follows: For each processing time, four experts were interviewed and these experts expressed their valuations in the form of TFNs. Table 1 gives the expert's TFN data and associated ordinary number for each job.

Table 1. The expert's TFN and associated ordinary number for each processing time

Job	Machine 1		Machine 2		Machine 3	
	Expert's TFN(P_{i1})	(P_{i1})	Expert's TFN(P_{i2})	(P_{i2})	Expert's TFN(P_{i3})	(P_{i3})
1	(5, 7, 9) (6, 7, 11)* (4, 8, 13)* (3, 6, 14)	7 7.75 8.25 7.25	(4, 8, 15)* (2, 11, 12) (3, 13, 13)* (1, 10, 15)	8.75 9 10.5 9	(13, 15, 21) (13, 16, 18)* (12, 20, 24)* (10, 20, 22)	16 15.75 19 18
2	(3, 8, 8)* (4, 5, 11) (2, 7, 8) (3, 4, 12)*	9.75 11.75 8.75 10.5	(8, 13, 17)* (6, 11, 21) (5, 10, 22) (9, 9, 10)*	12.75 12.25 11.75 9.25	(5, 7, 9)* (5, 6, 10) (5, 5, 5)* (5, 5, 5)*	7 6.75 5 5
3	(5, 11, 12) (6, 13, 15)* (4, 10, 11) (5, 12, 13)*	9.75 11.75 8.75 10.5	(4, 4, 4) (4, 6, 7)* (5, 5, 5) (5, 6, 7)*	4 5.75 5 6	(8, 10, 15) (8, 9, 12) (9, 10, 13)* (8, 11, 14)*	10.75 9.50 10.50 11
4	(6, 12, 13)* (5, 11, 14) (5, 10, 17) (2, 8, 10)*	10.75 10.25 10.50 7	(11, 13, 14)* (10, 12, 17) (10, 10, 10)* (8, 9, 18)	12.75 12.75 10 11	(6, 6, 6) (6, 6, 6) (7, 7, 7)* (5, 5, 5)*	6 6 7 5

The upper* means the major TFN for the pessimistic sequence, and the lower* means the minor TFN for the optimistic sequence.

For each job in Table 1, row 1 gives the TFN of expert 1, row 2 that for expert 2, etc. Using these data, a pessimistic sequence with major TFNs and an optimistic sequence with minor TFNs are now computed.

6.1 The Pessimistic Sequence

[Step 1−Step 6]

The pessimistic processing times composed of major TFNs are shown in Table 2.

Table 2. Pessimistic TFN for each processing time

Job	Machine 1		Machine 2		Machine 3	
	Expert's TFN($\tilde{P}_{1i}$)	($\hat{P}_{1i}$)	Expert's TFN($\tilde{P}_{2i}$)	($\hat{P}_{2i}$)	Expert's TFN($\tilde{P}_{3i}$)	($\hat{P}_{3i}$)
1	(4, 8, 13)	8.25	(3, 13, 13)	10.5	(12, 20, 24)	19
2	(3, 8, 8)	6.75	(8, 13, 17)	12.75	(5, 7, 9)	7
3	(6, 13, 15)	11.75	(5, 6, 7)	6	(8, 11, 14)	11
4	(6, 12, 13)	10.75	(11, 13, 14)	12.75	(7, 7, 7)	7

[Step 7 − Step 11]

1) The first branching is shown in Figure 1.

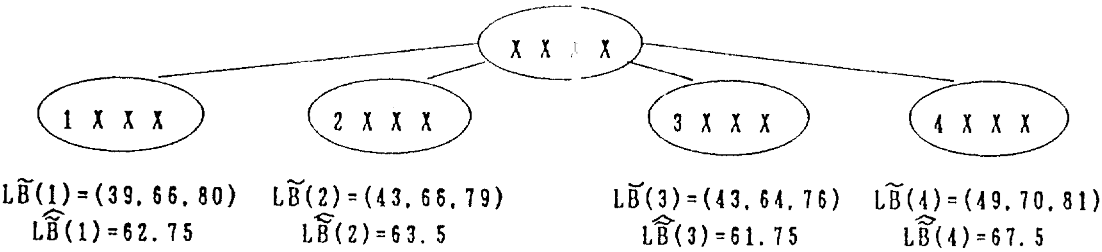

Figure 1. The First branching tree for a pessimistic solution

Case 1 : $\sigma=\{1\}$

$$\begin{aligned} lb_1 &= LC_1(1)+\{P_{12}+P_{13}+P_{14}\} + \min\{P_{22}+P_{32},\ P_{23}+P_{33},\ P_{24}+P_{34}\} \\ &= (4,8,13)+(15,33,36)+\min\{(13,20,26),\ (13,17,21),\ (18,20,21)\} \\ &= (4,8,13)+(15,33,36)+(13,17,21) \\ &= (32,58,70) \end{aligned}$$

$$\begin{aligned} lb_2 &= LC_2(1)+\{P_{22}+P_{23}+P_{24}\} + \min\{P_{32},\ P_{33},\ P_{34}\} \\ &= LC_1(1)+P_{21}+\{P_{22}+P_{23}+P_{24}\} + \min\{P_{32},\ P_{33},\ P_{34}\} \\ &= (31,53,64)+\min(5,7,9),\ (8,11,14),\ (7,7,7)\} \\ &= (31,\ 53,\ 64)+(7,7,7) \\ &= (38,60,71) \end{aligned}$$

$$\tilde{lb}_3 = \tilde{LC}_3(1) + \{\tilde{P}_{32} + \tilde{P}_{33} + \tilde{P}_{34}\}$$

$$= \tilde{LC}_2(1) + \tilde{P}_{31} + \{\tilde{P}_{32} + \tilde{P}_{33} + \tilde{P}_{34}\}$$

$$= (31,66,80)$$

$$\tilde{LB}(1) = \max\{\tilde{lb}_1,\ \tilde{lb}_2,\ \tilde{lb}_3\}$$

$$= \max\{(32,58,70),\ (38,60,71),\ (31,66,80)\}$$

$$= (31,66,80)$$

<u>Case 2 : $\sigma=\{2\}$</u>

$$\tilde{lb}_1 = \tilde{LC}_1(2) + \{\tilde{P}_{11} + \tilde{P}_{13} + \tilde{P}_{14}\} + \min\{\tilde{P}_{21} + \tilde{P}_{31},\ \tilde{P}_{23} + \tilde{P}_{33},\ \tilde{P}_{24} + \tilde{P}_{34}\}$$

$$= (19,41,49) + \min\{(15,33,37),\ (13,17,21),\ (18,20,21)\}$$

$$= (19,41,49) + (13,17,21)$$

$$= (32,58,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(2) + \{\tilde{P}_{21} + \tilde{P}_{23} + \tilde{P}_{24}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{33},\ \tilde{P}_{34}\}$$

$$= \tilde{LC}_1(2) + \tilde{P}_{22} + \{\tilde{P}_{21} + \tilde{P}_{23} + \tilde{P}_{24}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{33},\ \tilde{P}_{34}\}$$

$$= (37,60,66)$$

$$\tilde{lb}_3 = \tilde{LC}_3(2) + \{\tilde{P}_{31} + \tilde{P}_{33} + \tilde{P}_{34}\}$$

$$= \tilde{LC}_2(2) + \tilde{P}_{32} + \{\tilde{P}_{31} + \tilde{P}_{33} + \tilde{P}_{34}\}$$

$$= (43,66,79)$$

$$\tilde{LB}(2) = \max\{\tilde{lb}_1,\ \tilde{lb}_2,\ \tilde{lb}_3\}$$

$$= \max\{(32,58,70),\ (37,60,66),\ (43,66,78)\}$$

$$= (43,66,79)$$

<u>Case 3 : $\sigma=\{3\}$</u>

$$\tilde{lb}_1 = \tilde{LC}_1(3) + \{\tilde{P}_{11} + \tilde{P}_{12} + \tilde{P}_{14}\} + \min\{\tilde{P}_{21} + \tilde{P}_{31},\ \tilde{P}_{22} + \tilde{P}_{32},\ \tilde{P}_{24} + \tilde{P}_{34}\}$$

$$= (37,61,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(3) + \{\tilde{P}_{21} + \tilde{P}_{22} + \tilde{P}_{24}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{32},\ \tilde{P}_{34}\}$$

$$= \tilde{LC}_1(3) + \tilde{P}_{23} + \{\tilde{P}_{21} + \tilde{P}_{22} + \tilde{P}_{24}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{32},\ \tilde{P}_{34}\}$$

$$= (40,65,73)$$

$$\tilde{lb}_3 = \tilde{LC}_3(3)+\{\tilde{P}_{31}+\tilde{P}_{32}+\tilde{P}_{34}\}$$

$$= \tilde{LC}_2(3)+\tilde{P}_{33}+\{\tilde{P}_{31}+\tilde{P}_{32}+\tilde{P}_{34}\}$$

$$= (43,64,76)$$

$$\tilde{LB}(3) = \max\{(37,61,70),\ (40,65,73),\ (43,64,76)\}$$

$$=(43,64,76)$$

<u>Case 4 : $\sigma=\{4\}$</u>

$$\tilde{lb}_1 = \tilde{LC}_1(4)+\{\tilde{P}_{11}+\tilde{P}_{12}+\tilde{P}_{13}\} + \min\{\tilde{P}_{21}+\tilde{P}_{31},\ \tilde{P}_{22}+\tilde{P}_{32},\ \tilde{P}_{23}+\tilde{P}_{33}\}$$

$$= (32,58,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(4)+\{\tilde{P}_{21}+\tilde{P}_{22}+\tilde{P}_{23}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{32},\ \tilde{P}_{33}\}$$

$$= \tilde{LC}_1(4)+\tilde{P}_{24}+\{\tilde{P}_{21}+\tilde{P}_{22}+\tilde{P}_{23}\} + \min\{\tilde{P}_{31},\ \tilde{P}_{32},\ \tilde{P}_{33}\}$$

$$= (38,64,73)$$

$$\tilde{lb}_3 = \tilde{LC}_3(4)+\{\tilde{P}_{31}+\tilde{P}_{32}+\tilde{P}_{33}\}$$

$$= \tilde{LC}_2(4)+\tilde{P}_{34}+\{\tilde{P}_{31}+\tilde{P}_{32}+\tilde{P}_{33}\}$$

$$= (49,70,81)$$

$$\tilde{LB}(4) = \max\{(32,58,70),\ (38,64,73),\ (49,70,81)\}$$

$$= (49,\ 70,\ 81)$$

2) The second branching is shown in Figure 2

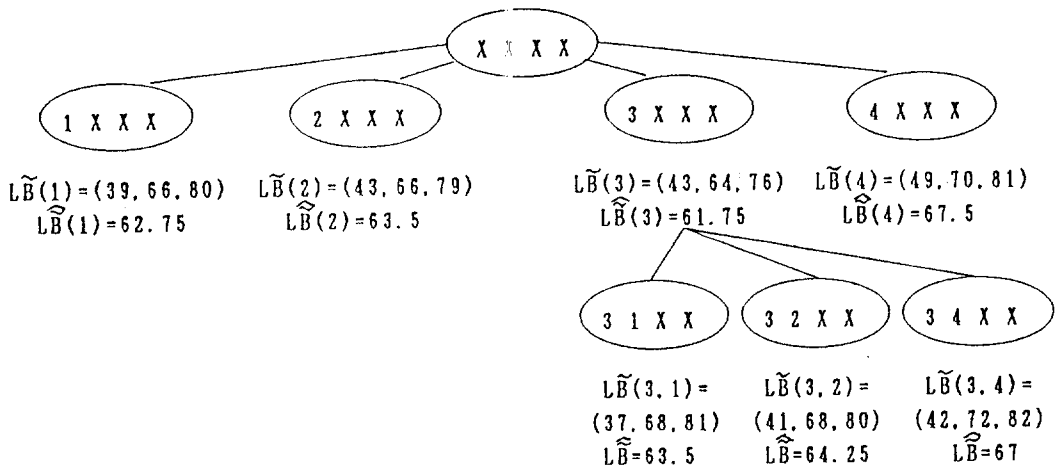

Figure 2. The second branching tree for a pessimistic solution

Case 1 : $\sigma=\{3,1\}$

$$\tilde{lb}_1 = \tilde{LC}_1(3,1)+\{\tilde{P}_{12}+\tilde{P}_{14}\}+\min\{\tilde{P}_{22}+\tilde{P}_{32},\ \tilde{P}_{21}+\tilde{P}_{31}\}$$
$$= (37,61,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(3,1)+\{\tilde{P}_{22}+\tilde{P}_{24}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{34}\}$$
$$= \max\{\tilde{LC}_1(3,1)+\tilde{P}_{21},\ \tilde{LC}_2(3)+\tilde{P}_{21}\}+\{\tilde{P}_{22}+\tilde{P}_{24}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{34}\}$$
$$= (39,67,79)$$

$$\tilde{lb}_3 = \tilde{LC}_3(3,1)+\{\tilde{P}_{32}+\tilde{P}_{34}\}$$
$$= \max\{\tilde{LC}_2(3,1)+\tilde{P}_{31},\ \tilde{LC}_3(3)+\tilde{P}_{31}\}+\{\tilde{P}_{32}+\tilde{P}_{34}\}$$
$$= (37,68,81)$$

$$\tilde{LB}(3,1) = \max\{(37,61,70),\ (39,67,79),\ (37,68,81)\}$$
$$= (37,68,81)$$

Case 2 : $\sigma=\{3,2\}$

$$\tilde{lb}_1 = \tilde{LC}_1(3,2)+\{\tilde{P}_{11}+\tilde{P}_{14}\}+\min\{\tilde{P}_{21}+\tilde{P}_{31},\ \tilde{P}_{24}+\tilde{P}_{34}\}$$
$$= (37,61,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(3,2)+\{\tilde{P}_{21}+\tilde{P}_{24}\}+\min\{\tilde{P}_{31}+\tilde{P}_{34}\}$$
$$= \max\{\tilde{LC}_1(3,2)+\tilde{P}_{22},\ \tilde{LC}_2(3)+\tilde{P}_{22}\}+\{\tilde{P}_{21}+\tilde{P}_{24}\}+\min\{\tilde{P}_{31}+\tilde{P}_{34}\}$$
$$= (38,67,74)$$

$$\tilde{lb}_3 = \tilde{LC}_3(3,2)+\{\tilde{P}_{31}+\tilde{P}_{34}\}$$
$$= \max\{\tilde{LC}_2(3,2)+\tilde{P}_{32},\ \tilde{LC}_3(3)+\tilde{P}_{32}\}+\{\tilde{P}_{31}+\tilde{P}_{34}\}$$
$$= (41,68,80)$$

$$\tilde{LB}(3,2) = \max\{(37,61,70),\ (8,67,74),\ (41,68,80)\}$$
$$= (41,68,80)$$

Case 3 : $\sigma=\{3,4\}$

$$\tilde{lb}_1 = \tilde{LC}_1(3,4)+\{\tilde{P}_{11}+\tilde{P}_{12}\}+\min\{\tilde{P}_{21}+\tilde{P}_{31},\ \tilde{P}_{22}+\tilde{P}_{32}\}$$
$$= (32,61,75)$$

$lb_2 = LC_2(3,4) + \{P_{21} + P_{22}\} + \min\{P_{31},\ P_{32}\}$

$= \max\{LC_1(3,4) + P_{21},\ LC_2(3) + P_{21}\} + \{P_{21} + P_{22}\} + \min\{P_{31},\ P_{32}\}$

$= (39,71,81)$

$lb_3 = LC_3(3,4) + \{P_{31} + P_{32}\}$

$= \max\{LC_2(3,4) + P_{31},\ LC_3(3) + P_{31}\} + \{P_{31} + P_{32}\}$

$= (47,72,82)$

$\widetilde{LB}(3,4) = \max\{(32,61,75),\ (39,71,81),\ (42,72,82)\}$

$= (42,72,82)$

3) The third branching is shown in Figure 3.

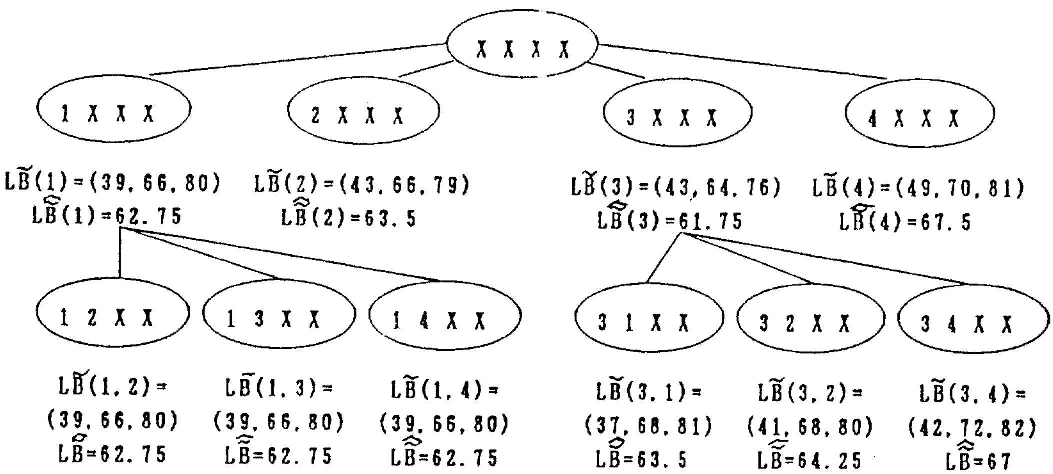

Figure 3. The third branching tree for a pessimistic solution

Case 1 : $\sigma = \{1,2\}$

$lb_1 = LC_1(1,2) + \{P_{13} + P_{14}\} + \min\{P_{23} + P_{33},\ P_{24} + P_{34}\}$

$= LC_1(1) + P_{12} + \{P_{13} + P_{14}\} + \min\{P_{23} + P_{33},\ P_{24} + P_{34}\}$

$= (32,58,70)$

$$
\begin{aligned}
\tilde{lb}_2 &= \tilde{LC}_2(1,3,2)+\tilde{P}_{24}+\tilde{P}_{34} \\
&= \max\{\tilde{LC}_1(1,3,2)+\tilde{P}_{22},\ \tilde{LC}_2(1,3)+\tilde{P}_{22}\}+\tilde{P}_{24}+\tilde{P}_{34} \\
&= (39,62,74)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{lb}_3 &= \tilde{LC}_3(1,3,2)+\tilde{P}_{34} \\
&= \max\{\tilde{LC}_2(1,3,2)+\tilde{P}_{32},\ \tilde{LC}_3(1,3)+\tilde{P}_{32}\}+\tilde{P}_{34} \\
&= (39,66,80)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{LB}(1,3,2) &= \max\{(37,61,70),\ (39,62,74),\ (39,66,80)\} \\
&= (39,66,80)
\end{aligned}
$$

Case 2 : $\sigma=\{1,3,4\}$

$$
\begin{aligned}
\tilde{lb}_1 &= \tilde{LC}_1(1,3,4)+\tilde{P}_{12}+\{\tilde{P}_{22}+\tilde{P}_{32}\} \\
&= \tilde{LC}_1(1,3)+\tilde{P}_{14}+\tilde{P}_{12}+\{\tilde{P}_{22}+\tilde{P}_{32}\} \\
&= (32,61,75)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{lb}_2 &= \tilde{LC}_2(1,3,4)+\tilde{P}_{22}+\tilde{P}_{32} \\
&= \max\{\tilde{LC}_1(1,3,4)+\tilde{P}_{24},\ \tilde{LC}_2(1,3)+\tilde{P}_{24}\}+\tilde{P}_{22}+\tilde{P}_{32} \\
&= (40,66,81)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{lb}_3 &= \tilde{LC}_3(1,3,4)+\tilde{P}_{32} \\
&= \max\{\tilde{LC}_2(1,3,4)+\tilde{P}_{34},\ \tilde{LC}_3(1,3)+\tilde{P}_{34}\}+\tilde{P}_{32} \\
&= (39,66,80)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{LB}(1,3,4) &= \max\{(32,61,75),\ (40,66,81),\ (39,66,80)\} \\
&= (40,66,81)
\end{aligned}
$$

The sequence which yields the lowest lower bound for the entire makespan is 1−3−2−4 with a fuzzy lower bound for the fuzzy makespan of (39,66,80). The fuzzy makespan of this sequence is

$$\tilde{m}_s = \max \tilde{c}_{3i(k)} = \tilde{c}_{34(4)} = (37,66,291).$$

The fuzzy mean flow time is then (26.25, 54.5, 155). The fuzzy parameters for this optimal sequence are listed in Table 3.

$lb_3 = LC_3(1,4) + \{P_{32} + P_{33}\}$

$= \max\{LC_2(1,4) + P_{34},\ LC_3(1) + P_{34}\} + \{P_{32} + P_{33}\}$

$= (39,66,80)$

$LB(1,2) = \max\{(32,58,70),\ (39,59,73),\ (39,66,80)\}$

$= (39,66,80)$

4) The fourth branching is shown in Figure 4.

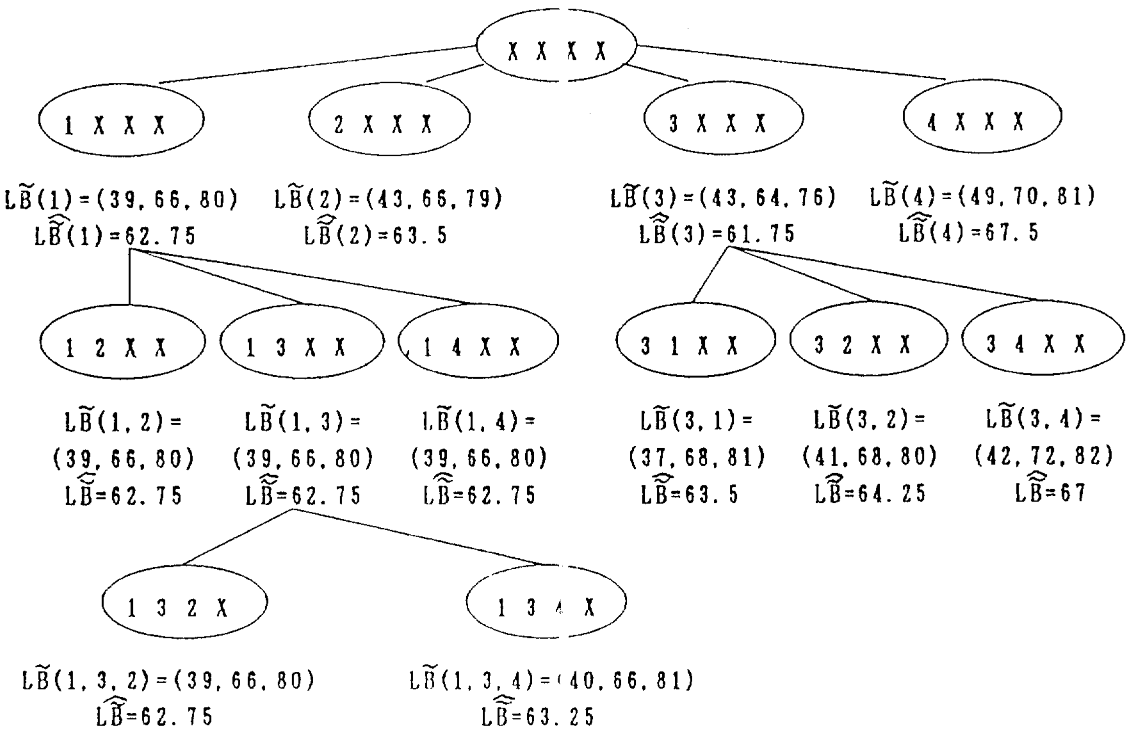

Figure 4. The fourth branching tree for a pessimistic solution

Case 1 : $\sigma = \{1,3,2\}$

$lb_1 = LC_1(1,3,2) + P_{11} + \{P_{21} + P_{31}\}$

$= LC_1(1,3) + P_{12} + P_{11} + \{P_{21} + P_{31}\}$

$= (37,61,70)$

$$\tilde{lb}_2 = \tilde{LC}_2(1,2)+\{\tilde{P}_{23}+\tilde{P}_{24}\}+\min\{\tilde{P}_{33},\ \tilde{P}_{34}\}$$

$$= \max\{\tilde{LC}_1(1,2)+\tilde{P}_{22},\ \tilde{LC}_2(1)+\tilde{P}_{22}\}+\{\tilde{P}_{23}+\tilde{P}_{24}\}+\min\{\tilde{P}_{33},\ \tilde{P}_{34}\}$$

$$= (38,60,71)$$

$$\tilde{lb}_3 = \tilde{LC}_3(1,2)+\{\tilde{P}_{33}+\tilde{P}_{34}\}$$

$$= \max\{\tilde{LC}_2(1,2)+\tilde{P}_{32},\ \tilde{LC}_3(1)+\tilde{P}_{32}\}+\{\tilde{P}_{33}+\tilde{P}_{34}\}$$

$$= (39,66,80)$$

$$\tilde{LB}(1,2) = \max\{(32,58,70),\ (38,60,71),\ (39,66,80)\}$$

$$= (39,66,80)$$

Case 2 : $\sigma=\{1,3\}$

$$\tilde{lb}_1 = \tilde{LC}_1(1,3)+\{\tilde{P}_{12}+\tilde{P}_{14}\}+\min\{\tilde{P}_{22}+\tilde{P}_{32},\ \tilde{P}_{24}+\tilde{P}_{34}\}$$

$$= \tilde{LC}_1(1)+\tilde{P}_{13}+\{\tilde{P}_{12}+\tilde{P}_{14}\}+\min\{\tilde{P}_{22}+\tilde{P}_{32},\ \tilde{P}_{24}+\tilde{P}_{34}\}$$

$$= (37,61,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(1,3)+\{\tilde{P}_{22}+\tilde{P}_{24}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{34}\}$$

$$= \max\{\tilde{LC}_1(1,3)+\tilde{P}_{23},\ \tilde{LC}_2(1)+\tilde{P}_{23}\}+\{\tilde{P}_{22}+\tilde{P}_{24}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{34}\}$$

$$= (41,60,73)$$

$$\tilde{lb}_3 = \tilde{LC}_3(1,3)+\{\tilde{P}_{32}+\tilde{P}_{34}\}$$

$$= \max\{\tilde{LC}_2(1,3)+\tilde{P}_{33},\ \tilde{LC}_3(1)+\tilde{P}_{33}\}+\{\tilde{P}_{32}+\tilde{P}_{34}\}$$

$$= (39,66,80)$$

$$\tilde{LB}(1,3) = \max\{(37,61,70),\ (41,60,73),\ (39,66,80)\}$$

$$= (39,66,80)$$

Case 3 : $\sigma=\{1,4\}$

$$\tilde{lb}_1 = \tilde{LC}_1(1,4)+\{\tilde{P}_{12}+\tilde{P}_{13}\}+\min\{\tilde{P}_{22}+\tilde{P}_{32},\ \tilde{P}_{23}+\tilde{P}_{33}\}$$

$$= (32,58,70)$$

$$\tilde{lb}_2 = \tilde{LC}_2(1,4)+\{\tilde{P}_{22}+\tilde{P}_{23}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{33}\}$$

$$= \max\{\tilde{LC}_1(1,4)+\tilde{P}_{24},\ \tilde{LC}_2(1)+\tilde{P}_{24}\}+\{\tilde{P}_{22}+\tilde{P}_{23}\}+\min\{\tilde{P}_{32},\ \tilde{P}_{33}\}$$

$$= (39,59,73)$$

Table 3. Fuzzy parameters of pessimistic optimal sequence

i	k	$q_{1(k)}$	$p_{1(k)}$	$c_{1(k)}$	$q_{2(k)}$	$p_{2(k)}$	$c_{2(k)}$	$q_{3(k)}$	$p_{3(k)}$	$c_{3(k)}$
1	1	0	(4, 8,13)	(4, 8,13)	0	([illegible],13,13)	(7,21, 26)	0	(12,20,24)	(19,41, 50)
3	2	(4, 8,13)	(6,13,15)	(10,21,28)	(0, 0,16)	([illegible], 6, 7)	(15,27, 51)	(0,14,35)	(8,11,14)	(23,52,100)
2	3	(10,21,28)	(3, 8, 8)	(13,29,36)	(0, 0,38)	([illegible],13,17)	(21,42, 91)	(0,10,79)	(5, 7, 9)	(26,59,179)
4	4	(13,29,36)	(6,12,13)	(19,41,49)	(0, 1,72)	(1[illegible],13,14)	(30,55,135)	(0, 4,149)	(7, 7, 7)	(37,66,291)

6.2 The Optimistic Sequence with Minor TFNs

[Step 1－Step 6]

The optimistic processing times composed of minor TFNs are shown in Table 4.

Table 4. Optimistic TFN for each processing time

Job	Machine 1		Machine 2		Machine 3	
	Expert's TFN(P_{1i})	$(\hat{P}_{1i})$	Expert's TFN(P_{2i})	$(\hat{P}_{2i})$	Expert's TFN(P_{3i})	$(\hat{P}_{3i})$
1	(5, 7, 9)	7	(4, 8, [illegible]5)	8.75	(13,16,18)	15.75
2	(3, 4,12)	5.75	(9, 9, [illegible]0)	9.25	(5, 5, 5)	5
3	(4,10,11)	8.75	(4, 4, [illegible]4)	4	(8, 9,12)	9.50
4	(2, 8,10)	7	(10,10, [illegible]0)	10	(5, 5, 5)	5

[Setp 7－Step 11]

1) The first branching is shown in Figure 5.

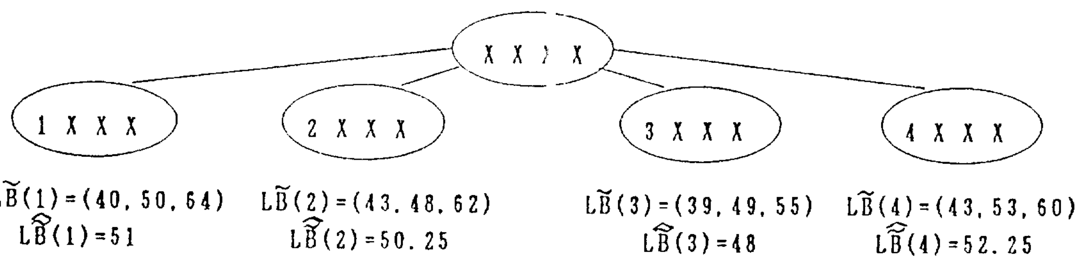

Figure 5. The first branching tree for an optimistic solution

2) The second branching is shown in Figure 6.

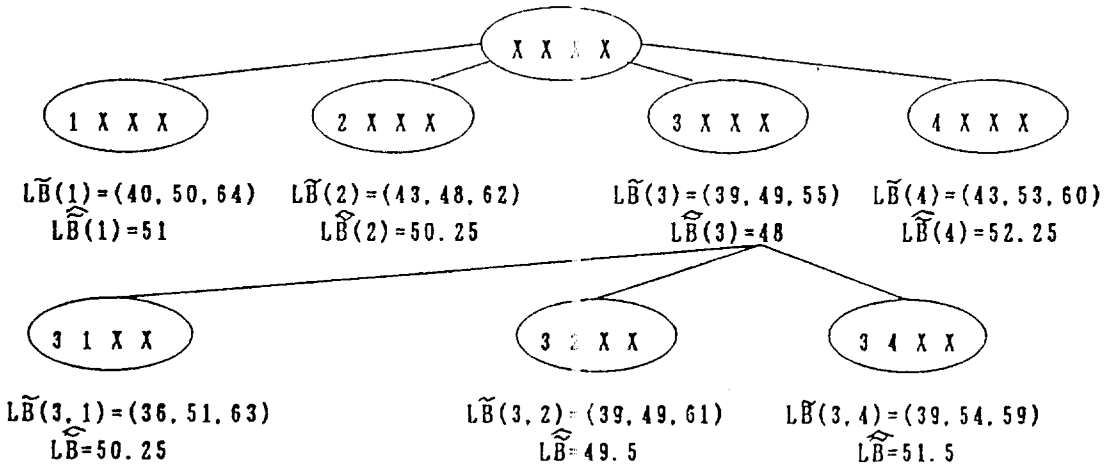

Figure 6. The second branching tree for an optimistic solution

3) The third branching is shown in Figure 7.

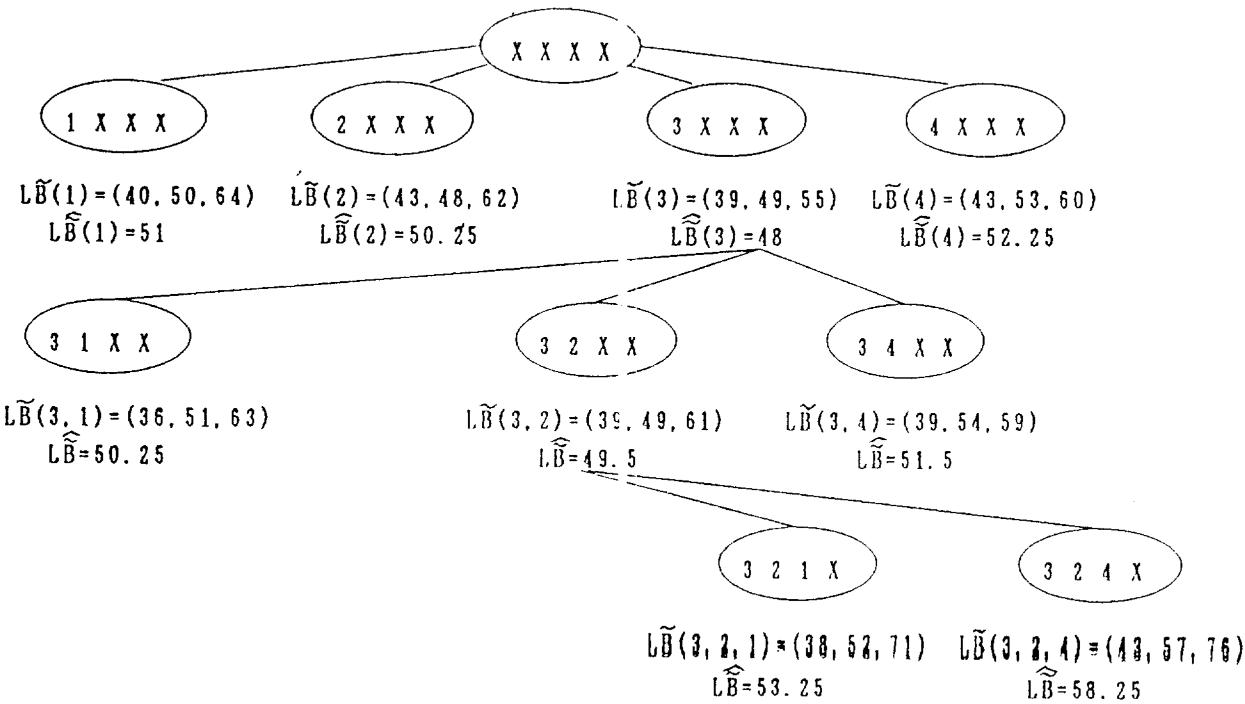

Figure 7. The third branching tree for an optimistic solution

4) The fourth branching is shown in Figure 8.

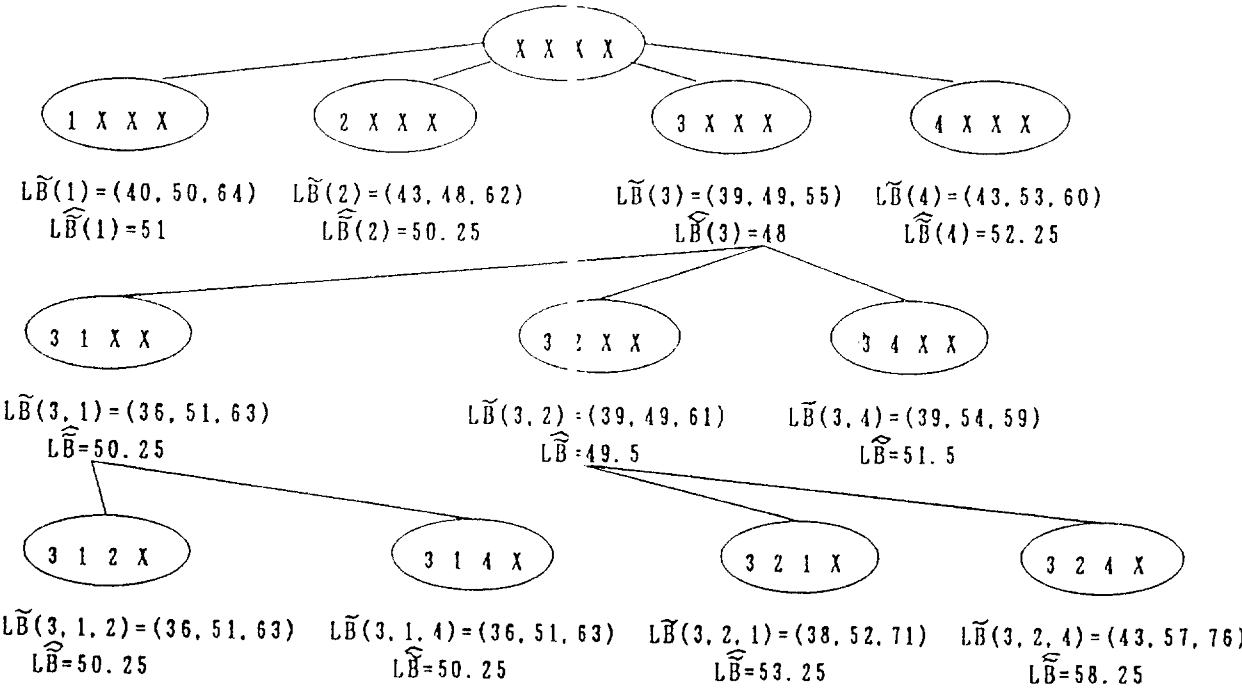

Figure 8. The fourth branching tree for an optimistic solution

Fuzzified branch & bound algorithm yields then optimal job sequence of 3−1−4−2. The fuzzy lower bound for the fuzzy makespan is (36,51,63)(N.B. Schedules 3−1−2−4 and 3−1−4−2 are both optimal with (36,51,63) but, the fuzzy makespan of 3−1−2−4 is worse than the fuzzy makespan of 3−1−4−2).

The fuzzy makespan of this sequence is

$$\tilde{m}_s = \max\ \tilde{c}_{3(1)(4)},\ \tilde{c}_{3(2)(4)} = (28,51,217).$$

The fuzzy mean flow time is then (24.10,25,111). The fuzzy parameters of the optimistic optimal sequence are listed in Table 5.

What is gained by using the TFN representation is the range of values for m_F and m_s which are truly TFNs. The decision maker now knows the spread of the m_F and m_s, whereas it was assumed away and lost in the deterministic simplification. Accordingly, the manager can now

Table 5. Fuzzy parameters of optimistic optimal sequence

i	k	$q_{1[k]}$	$p_{1[k]}$	$C_{1[k]}$	$q_{2[k]}$	$p_{2[k]}$	$C_{2[k]}$	$q_{3[k]}$	$p_{3[k]}$	$C_{3[k]}$
1	1	0	(4,10,11)	(4,10,11)	0	(4, 4, 4)	(8,14,15)	0	(8, 9,12)	(16,23, 27)
3	2	(4,10,11)	(5, 7, 9)	(9,17,20)	(0,0, 6)	(4, 8,15)	(13,25,41)	(0,0, 14)	(13,16,18)	(26,41, 73)
2	3	(9,17,20)	(2,8,10)	(11,25,30)	(0,0,30)	(10,10,10)	(21,35,70)	(0,6, 52)	(5, 5, 5)	(26,46,127)
4	4	(11,25,30)	(3,4,12)	(14,29,42)	(0,6,56)	(9, 9,10)	(23,44,108)	(0,2,104)	(5, 5, 5)	(28,51,217)

plan. In addition, the experts can re-evaluate their TFN processing times with the pessimistic and optimistic sequences.

4. Conclusions

In this paper, we proposed a pessimistic sequence with major data and an optimistic sequence with minor data. The direct use of fuzzy numbers to modeling imprecision was emphasized in this paper. In particular, we use only TFNs in our example. Although general fuzzy numbers can be used to represent the uncertain processing times, the increase in computational effort by the use of a general fuzzy number is tremendous. Furthermore, since the fuzzy value is only an approximate estimate, it is very difficult to estimate a general fuzzy number representation of this processing time.

One of the problems in solving job scheduling problems by considering fuzzy numbers is comparing the fuzzy numbers to obtain a total order. Many people have studied the comparison of fuzzy numbers and have proposed several methods. None of the existing methods are perfect. The decision maker should choose the comparison method based upon the goals of the fuzzy application.

In the example the data were established by four experts as TFNs, but in a practical example the number of experts depends on many factors: available experts, kind of job, cost of operation, gravity of scheduling and so on. However, it appears that the computations of the major TFN and minor TFN are very useful in many realistic situations. There is no a priori rule for their selection. For each processing time, a major TFN and a minor TFN are selected. Of course, if only one TFN exists, it is its own major and minor. To determine the pessimistic sequence, the major TFNs are employed. For optimistic sequence, minor TFNs are employed.

With the calculations of the fuzzy parameters, the fuzzy makespan are then calculated. Following these calculations, the experts re-evaluate their TFN estimates and the process is repeated. The method described in this paper is useful especially when the opinions of many experts are required in a large scale job scheduling problem.

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