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Optimal Inspection Plan for a Flexible Assembly Line

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Abstract

This paper considers an inspection station location problem for a flexible assembly line that consists of multiple stages. By considering the trade-off between locating an station at an "early" stage and at a "late" stage, we have developed some dominance relations on a graph. Based on the dominance relations, we then present an algorithm that finds an optimal inspection plan and a heuristic approach that finds a near-optimal inspection plan. The effectiveness of these two algorithms is demonstrated by two numerical examples.

1. Introduction

Quality control is an important aspect in manufacturing. In order to achieve high quality levels in a production facility, manufacturers assign quality checkers(or inspection stations) to inspect finished and in-process products. The inspection station location problem confronted by manufacturers is to determine the number and location of inspection stations that will minimize the total cost of manufacturing a quality product.

In this paper we will investigate an optimal location of inspection stations for a flexible flow line consisting of a sequence of stages, each of which performs a separate set of operations. Each product is processed seqentially through every stage. To understand the trade off between the inspection and scrapping cost, consider the following situations :

1. One inspection station is assigned at the final stage. Then scrapping cost for defective items is high due to the wasted effort incurred at the early production stages.
2. An inspection station is assigned at every stage. In this situation, the total inspection cost is high.

The issue at hand therefore is to find the optimal trade off between extra inspection costs

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and unneccessary production costs due to the late detection of a defect. Since it is difficult to change the production line configuration once inspection devices are installed and the line flow is balanced, the problem addressed in this paper is important and is needed for planning prior to production [2,3,4,6].

A variant of the problem has been considered by White [7,8], in which a shortest path algorithm is employed to determine an optimal inspection configuration. As an alternative approach to solving this problem, in this paper dominance relations on a graph with a simple cost structure are used to reduce and circumvent the inherent computational burden.

The organization of this paper is as follows. Section 2 describes the model and its basic assumptions. Section 3 develops the theoretical framework used to analyze the model. Section 4 develops dominance relations that are employed in an optimum algorithm. A heuristic approach for the problem is presented at this time too. Computational results of both algorithms are then presented in section 5. Finally, in section 6, possible contributions of this paper are discussed and future research directions are suggested.

2. Model and Assumptions

In this section we consider a problem that determines an cost-efficient optimal location of inspection stations in a production line. The production line has N stages, each of which processes a separate set of operations. Let P_i be the probability that a non-defective item becomes defective at stage i. Assume that P_i is a known discrete value and is independent of the other stages. This assumption is reasonable as it models the situation where P_i represents the probability of machine malfunction or tool failure at processing stage i. The other basic assumptions for the model are following:

1. Inspection can be performed at the end of each stage. (This is realistic for a multistage production line.)
2. Inspection is perfect.
3. All defective items are non-repairable, and are discarded once discovered. (This is quite reasonable in the electronic industry as the cost of repairing defects is usually more expensive than the production cost incurred.)
4. There is always an inspection station located at the final stage. (If substandard and defective products reach the market place with any degree of frequency, the reputation of a compnay is placed at a risk.)

An example of a five stage production line that satisfies the above assumptions is depited in Figure 1.

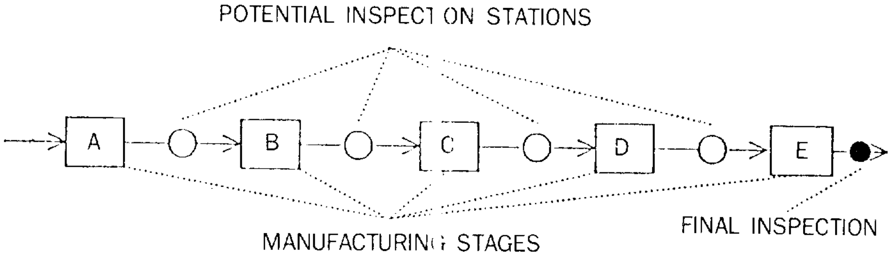


Figure 1. The Configur tion of a Product Line

It is noteworthy to mention that this kind of nanufacturing system is commonly used in electronic parts assembly. (See [9] for detailed descri tion of this kind of manufacturing system.)

With these assumptions, the following processi ng propety describes the transition of non-defective and defective products:

P_i = *Prob* { defective at the end of stage i | non-defective before stage i }

A processing stage can tranform a good produ t into a defective one, but a defective product will remain defective during subsequent process ng with probability 1. Thus the output of the ith manufacturing stage on an individual unit ca be represented by the transition matrix:

$$\begin{array}{c} \\ G \\ D \end{array} \begin{array}{c} \begin{array}{cc} G & D \end{array} \\ \begin{bmatrix} 1-P_i & P_i \\ 0 & 1 \end{bmatrix} \end{array}$$

Given that an inspection is made at the end f stage i, the cost incurred at stage i consists of two components, an inspection cost denoted by I_i and scrapping cost of a defective non-repairable product discovered at the end of stag i, denoted by C_i. It is reasonable to assume that C_i is increasing in i for $i=1,\cdots,N$, as valu is being added onto the product as it passes through the stages[1].

The following is a summary of the notation th at will be used in the next section to develop a theoretica framwork for the model.

1) As a product passes through stages, the production cost incurred is accumulated.

P_i : the probability that a non-defective item before stage i is defective at the end of stage i.

$P_{i,j}$: the probability that a non-defective item before stage i is discovered defective at the end of stage j for $1 \leq i \leq j \leq N$

$$i.e., \quad P_{i,j}=1-\prod_{n=i}^{j}(1-P_n) \text{ and } P_{i,i}=P_i \text{ by definition.}$$

C_i : the scrapping cost of one defective item that is discovered at the end of stage i.

I_i : the cost of inspecting one unit at the end of stage i.

Let $X=(x_1, x_2, \cdots, x_N)^T$ denote an inspection plan, where

$$x_i = \begin{cases} 1 & \text{if an inspection is made at the end of stage } i \\ 0 & \text{otherwise.} \end{cases}$$

Assumption (4) asserts that there is an inspection station at the final stage in all of the feasible plans through. Hence, each feasible plan can be written as follows:

$$X = (x_1, x_2, \cdots, x_{N-1}, 1)^T$$

For any given inspection plan X, the total expected cost (TEC) is denoted by $TEC(X)$ and is equal to the sum of the expected costs incurred between any two consecutive inspection stations as a result of Assumption (2). As an example, consider the inspection plan $X = (1,0,1)^T$ of a three stage manufacturing system: *i.e.*, inspect a product at the end of stages 1 and 3. The total expected cost of inspection plan X is the sum of the expected cost at stage 1 plus the expected cost between stage 2 and 3. It then follows that $TEC(X)$ can be expressed as follows:

$$TEC(X) = P_1 \cdot C_1 + I_1 + (1-P_1) \cdot (P_{2,3} \cdot C_3 + I_3)$$

The location problem of inspection stations can now be written as follows:

$$\begin{aligned} &\text{Min } TEC(X) \\ &\text{s.t. } \; X=(x_1, \cdots, x_{N-1}, 1)^T \qquad \text{(P)} \\ &\qquad x_i = 0 \text{ or } 1 \text{ for } i=1,\cdots,N-1. \end{aligned}$$

With the above assumptions and the cost structure of the model, the next section will develop a theoretical framework to help determine an optimal inspection plan X^* that minimizes the total expected cost (TEC).

3. Theoretical Framework

3.1 Analysis

Denote the set of feasible plans for problem (P') by S^*:

$$S^* = \{X \mid X = (x_1, x_2, \cdots, x_{N-1}, 1)^T, \text{ where } x_i = 0 \text{ or } 1\}$$

Let $I(X) = \{l \mid x_l = 1,\ 0 \le l \le N-1\}$, where $I(X)$ is the set of stages at which inspection stations are assigned. For notational convenience, let stage 0 always be an element of $I(X)$. Also let $n(i) = \min\{l \mid 0 \le i < l \le N,\ i \in I(X),\ x_l = 1\}$, where $n(i)$ is the next inspection station after stage i. Let $f(i, n(i))$ denote the total expected cost incurred from stage $(i+1)$ to $n(i)$ given that the input to stage $(i+1)$ is non-defective. Inspection cost $I_{n(i)}$ is incurred. A scrapping cost however is incurred only if there is a malfunction between stage $(i+1)$ and $n(i)$. Using the assumption of non-defective input to stage $(i+1)$, $f(i, n(i))$ can now be expressed as follows:

$$f(i, n(i)) = (1 - P_{1,i}) \cdot (P_{i+1, n(i)} \cdot C_{n(i)} + I_{n(i)}) \tag{1}$$

where $i \in I(X)$ and $P_{i,j} = 0$ if $i > j$.

Then, $TEC(X)$ can be represented as follows:

$$TEC(X) = \sum_{i \in I(X)} f(i, n(i)) \tag{2}$$

As an example, consider the inspection plan $X = (1,0,1,0,1)^T$ for a five stage manufacturing system. The total expected cost of X, $TEC(X)$, can be expressed in the following way.

$$TEC(X) = \sum_{i \in \{0,1,3\}} f(i, n(i)) \tag{3}$$

$$= P_1 \cdot C_1 + I_1 + (1 - P_1) \cdot (P_{2,3} \cdot C_3 \cdot I_3) + (1 - P_{1,3}) \cdot (P_{1,5} \cdot C_5 + I_5)$$

From Eq (2), $TEC(X)$ is only related to $I(X)$, the index of $f(i, n(i))$. It can be shown from Eqs (1) and (2) that $TEC(X)$ is nonlinear in the x_i's. Therefore a standard algorithm for solving the problem (P) is not applicable. If an enumerative method is used as a decision making tool, the number of feasible inspection plans increases by 2^{N-1} in the number of manufacturing stages N; complete enumeration is prohibited even for a moderate number of manufacturing stages.

The purpose of this paper is to develop an algorithm for the location problem of inspection stations with fixed number of manufacturing stages. The structure of the total expected cost for a feasible inspection plan could be a point to explore toward an efficient algorithm. As shown in Eq (2), $TEC(X)$ has a decomposition property because of the assumption of perfect inspection: if inspection is made at a stage, all undetected failures before that stage will not effect subsequent processing stages. In other words, $TEC(X)$ can be divided into independent $f(i, n(i))$ items according to the index $f(i,n(i))$, $I(X)$. For example, any two inspection plans X and Y can have the same $f(i,n(i))$s in both $TEC(X)$ and $TEC(Y)$. This simple observation provides a promising basis for deriving a special characteristics in relations of feasible plans. Define $S_{(i,k)} = \{X \mid X \in S^*,\ i \in I(X),\ n(i)=k\}$ for $0 \le i < k \le N$. $S_{(i,k)}$ is the set of inspection plans where the next inspection after stage i is at stage k. Let X^+ represent the augumented inspection plan of X in which only one extra inspection station is added to X. For example, for any $X \in S_{(i,k)}$ and $B(i)=\{l \mid i<l<n(i)\}$, $X^+ = X+e$ where

e is a N-tuple unit vector such that $e_i = \begin{cases} 1 & \text{for only one element of } j \in B(j) \\ 0 & \text{the others.} \end{cases}$

Similiarly, let X^- represent the decremental inspection plan of X in which only one inspection station is deleted from X. We call X and X^+ adjacent inspection plans, as are X and X^-. For any two inspection plans, X and Y, X is preferred to Y if $TEC(X)<TEC(Y)$, dentoed by $X \succ Y$. The decomposition property of $TEC(X)$ makes it easy to establish the relation between any two adjacent inspection plans.

Theorem 1. *For any inspection plan $X \in S_{(i,k)}$ where $0 \le i < k \le N$,*

$$X^+ \prec X \quad \textit{if and only if } C_k + I_k < C_j + \frac{I}{P_{ij}} \qquad \textit{where} \qquad j \in B(i) = \{l \mid i<l<k\}.$$

Proof. Given any $X \in S_{(i,k)}$ and $X^+ = X+e$ where

$$e_i = \begin{cases} 1 & \text{for only one element of } j \in B(j) \\ 0 & \text{the others} \end{cases} \qquad \text{for } I'(i) = \{l \mid i<l<k\},$$

it follows from the definitions of X^+ and $TEC(X)$ in Eq(2) that

$$TEC(X) = \sum_{p \in I(X)} f(p, n(p))$$

$$TEC(X^+) = \sum_{p \in I(X) \cup \{i\}} f(p, n(p)) + f(i, n(i)) - f(j, n(j)).$$

By the definitions, $n(i)=j$ and $n(j)=k$ for $1 \le i < j < k \le N$.

$$X' \prec X \iff TEC(X) < TEC(X') \iff f(i,k) < f(i,j) + f(j,k) \qquad (4)$$

It follows from the definition of $f(i, n(i))$ in Eq. (1) that

$$f(i,k) = (1-P_{1,i})\,(P_{i+1,k}\cdot C_k + I_k)$$
$$f(i,j) = (1-P_{1,i})\,(P_{i+1,j}\cdot C_j + I_j)$$
$$f(j,k) = (1-P_{1,j})\,(P_{j+1,k}\cdot C_k + I_k).$$

Hence (4) can be written as follows :

$$(1-P_{1,i})\cdot(P_{i+1,k}\cdot C_k+I_k) < (1-P_{1,i})\cdot(P_{i+1,j}\cdot C_j+I_j)+(1-P_{1,j})\cdot(P_{j+1,k}\cdot C_k+I_k)$$
$$\iff \{(1-P_{1,i})\cdot P_{i+1,k}-(1-P_{1,j})\cdot P_{j+1,k}\}\cdot C_k+\{(1-P_{1,i})-(1-P_{1,j})\}\cdot I_k<(1-P_{1,i})\cdot P_{i+1,j}\cdot C_j+(1-P_{1,i})\cdot I_j. \qquad (5)$$

Since

$$\begin{aligned}
&(1-P_{1,i})\cdot P_{i+1,k}-(1-P_{1,j})\cdot P_{j+1,k}\\
&=(1-P_{1,i})\cdot\{P_{i+1,k}-(1-P_{i+1,j})\cdot P_{j+1,k}\}\\
&=(1-P_{1,i})\cdot(P_{i+1,k}-P_{j+1,k}+P_{i+1,j}\cdot P_{j+1,k})\\
&=(1-P_{1,i})\cdot\{\{1-(1-P_{i+1})(1-p_{i+2})\cdots(1-P_k)\} - \{1-(1-P_{j+1})\cdots(1-P_k)\}\\
&\quad+\{1-(1-P_{i+1})\cdots(1-P_j)\}\cdot\{1-(1-P_{j+1})\cdots(1-P_k)\}\} = (1-P_{1,i})\cdot P_{i+1,j}
\end{aligned}$$

and

$$(1-P_{1,i})-(1-P_{1,j})=P_{1,j}-P_{1,i}=(1-P_{1,i})\cdot P_{i+1,j}.$$

(5) can be rewritten as follows:

$$(1-P_{1,i})\cdot P_{i+1,j}\cdot C_k+(1-P_{1,i})\cdot P_{i+1,j}\cdot I_k<(1-P_{1,i})\cdot P_{i+1,j}\cdot C_j+(1-P_{1,i})\cdot I_j \iff C_k+I_k<C_j+\frac{I_j}{P_{i+1,j}}.$$

Therefore, we conclude that $X \prec X' \iff C_k+I_k<C_j+\dfrac{I_j}{P_{i+1,j}}$. ■

Theorem 1 implies that, when we compare two adjacent inspection plans in terms of total expected cost, we need to make only a simple comparison between them. Moreover, the comparison involves only a few stages instead of all of the stages shown in Eqs (2) and (3). The decomposition property implies that other adjacent inspection plans with the same relation can exist. The following proposition states the result.

Proposition 1. *If there exists an inspection plan* X *s.t.* $X \in S_{(i,k)}$ *and* $X \succ X'$, *then* $Y \succ Y'$ *for any* $Y \in S_{(j,k)}$.

Proof. It follows from Theorem 1 that for $X \in S_{(i,k)}$, $X \succ X^{j}$ implies that

$$C_k + I_k < C_j + \frac{I_j}{P_{i-1j}} \qquad \text{for } 0 \le i < j < k \le N.$$

This can be written as follows :

$$P_{i+1j} \cdot (C_k + I_k - C_j) < I_j .$$

The RHS can be manipulated as follows :

$$P_{i+1j} \cdot (C_k + I_k - C_j) < (1 - P_{i+1j} + P_{i+1j}) \cdot I_j .$$

We rearrange the inequality as follows.

$$P_{i+1j} \cdot (C_k + I_k - C_j - I_j) < (1 - P_{i-1j}) \cdot I_j . \tag{6}$$

The LHS of (6) represents the reduction in the total expected cost that occurs by adding the inspection station at the stage j between stage i and k. On the other hand, the RHS of (6) represents the unnecessary expected inspection cost that occurs at the j^{th} stage as a result of the non-defective processings from the stage $i+1$ to j. Hence, (6) implies that the addition of the inspection station at the stage j between stage i and k produces the net expected loss rather over the original plan X. This relation holds for any $X \in S_{(i,k)}$.

Therefore, we can conclude that $Y \succ Y^{j}$ for any $Y \in S_{(i,k)}$. ■

With Proposition 1, we gain even a greater reduction in the complexity of the problem as we can use a known preference relationship to determine the preference directions on additional adjacent inspection plans. As implied by Theorem 1, the preference relationship can be obtained by relatively simple comparison. Because Theorem 1 and Proposition 1 imply a potential method for reducing the computational burden, it seems very promising to formulate the problem as a graph and to find the optimal inspection configuration.

3.2 Constructing the Graph

Graphs are widely used in many different disciplines to model a symmetric relationship between objects [5]. The objects are represented by the nodes of the graph and two objects are connected by an arc if the objects are related.

In order to employ Theorem 1 effectively, it is convenient to construct a graph (J, A), where J and A are the set of nodes and the set of arcs on the graph, respectively. Each node represents a feasible inspection plan on this graph :

$$J=\{X \mid X \in S^{*}\}.$$

For a given inspection plan X, X is linked to Y on this graph if $Y=X^{+}$ or X^{-}, which is an adjacent plan of X. Hence, the set of arcs A can be written as follows:

$$A=\{(X,\ Y) \mid X,\ Y \in S^{*},\ Y=X^{+}\ or\ X^{-}\}.$$

For notational convenience, the graph is arranged into N levels where i consists of nodes that represent inspection plans with i inspection stations. Therefore, the set of nodes at level i is written as

$$S_i=\{X \mid X \in S^{*},\ E^{T}X=i\}$$

where E is an N-tuple unit vector all of whose elements are 1. For example, $(0,1,1,\cdots,1,1) \in S_{N-1}$. Let X_i denote any node in the set of nodes at level i, S_i. Since S_1 has only one node, $S_1=X_1$ and likewise $S_N=X_N$. The nodes are arranged in lexicographical order[2] at the same level. Figure 2 represents the five stage manufacturing system shown in Figure 1.

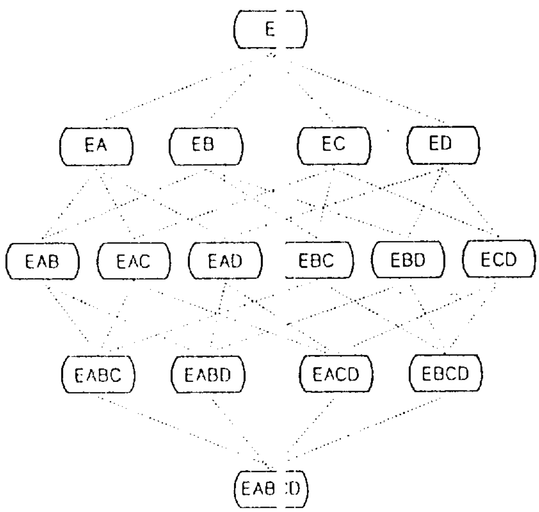


Figure 2. The Graph Construction of a Five Stage Production Line

2) A node $(x_1, x_2, \cdots, x_N)$ comes before a node $(y_1, y_2, \cdots, y_N)$, if either

1. $x_1>y_1$ or
2. $x_1=y_1$ and $x_2>y_2$ or

⋮

N. $x_1=y_1,\ x_2=y_2, \cdots, x_{N-1}=y_{N-1}$, and $x_N>y_N$.

We say that node X dominates Y if $X \succ Y$, and we represent the preference direction on the arc (X, Y) by $Y \rightarrow X$. In the graph (J, A), there are 2^{N-1} nodes and $(N-1) \cdot 2^{N-1}$ arcs. As we follow a path on this graph, the preference relationship on an arc is relatively simple to determine by using Theorem 1. Once the preference direction is determined, the dominated node cannot be the optimal plan; therefore only dominating nodes and their incident arcs are considered for further search. We can use Proposition 1 to further refine the search by using its preference relation to determine the preference on other arcs on (J, A). Therefore, the number of nodes that need to be evaluated is reduced.

4. Algorithms

4.1 Extension of Dominance Relation

Applying Theorem 1 and Proposition 1 to the graph (J, A), we can reduce the burden of compuational difficulty. To be more efficient, we need a systematic procedure to find an optimal inspection plan. As the first step, we develop some properties that can be used in pruning more dominated nodes at the begining of any path.

Define E_j as an N-tuple unit vector such that

$$e_i = \begin{cases} 1 & \text{for only the } j\text{th element} \\ 0 & \text{the others} \end{cases} \qquad j=1,2,\cdots,N-1$$

X_N is defined as the inspection plan that assigns an inspection station at every stage, and X_1 is the plan that has only one inspection station at the final stage. The following propositions state the results.

Proposition 2. *If $X_N \succ X_N - E_j$ for $j=1,2,\cdots,N-1$, then $Y \prec Y+E_j$ for any $Y \in S_{(i,j+1)}$ and $0 \le i < j < N$.*

Proof. Assume $X_N \succ X_N - E_j$. Then Theorem 1 says that

$$C_{j+1} + I_{j+1} > C + \frac{I_j}{P_{j,j}} \tag{7}$$

Since $0 \le (1-P_i) \le 1$, $1-(1-P_1)(1-P_2)\cdots(1-P_j) \ge 1-(1-P_2)\cdots(1-P_j) \ge \cdots \ge 1-(1-P_j)$. Hence, $P_{k,j}$ is decreasing in k, and $C_j + \frac{I_j}{P_{k,j}}$ is increasing in k.

Thus, (7) can be extended as follows:

$$C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}}>C_j+\frac{I_j}{P_{j-1j}}>\cdots>C_j+\frac{I_j}{P_{1j}}\ . \tag{8}$$

(8) can be written separately as follows:

$$C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}},\cdots,C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{1j}}\ . \tag{9}$$

By Proposition 1, these inequalities in (9) imply that

for any $Y\in S_{ij-1}$ $\quad Y\prec Y+E_j \quad 0\le i<j<N$. ■

By symmetry of the graph, we have the following corollary.

Corollary 1. *If* $X_1\succ X_1+E_j$ *for* $j=1,2,\cdots,N-1$, *then* $Y\succ Y+E_j$ *for any* $Y\in S_{(i,N)}$ *and* $0\le i<j<N$.

With Proposition 2, we can further reduce the number of nodes that will eventually be evaluated, and we can prune the arcs systematically from either X_1 or X_N.

We can extend the property developed in Proposition 2 by making an additional assumption about the cost structure. Assume that C_i+I_i is increasing in i. Since we can expect that I_i is usually less than the scrapping cost difference $C_{i+1}-C_i$, this assumption is quite reasonable. This assumption can be used to strengthen Proposition 2 by pruning additional dominated nodes at the beginning of any path. Let $S(j)$ represent the set of inspection plans where each plan has an inspection station at stage j for $j=1,2,\cdots,N$. For example, $S(1)$ is the set of inspection plans which have an inspection station at the end of stage 1.

Proposition 3. *If* $X_N\succ X_N-E_j$ *for* $j=1,2,\cdots,N-1$ *and* C_i+I_i *is increasing in* i, *then* $Y\succ Y-E_j$ *for any* $Y\in S(j)$ *and* $j=1,2,\cdots,N-1$.

Proof. Recall from Theorem 1 that if $X_N\succ X_N-E_j$, then

$$C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}}\ .$$

Also note that $C_j+\frac{I_j}{P_{kj}}$ is increasing in k. Hence,

$$C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}}>\cdots>C_j+\frac{I_j}{P_{1j}}\ . \tag{10}$$

Combining (10) with the assumption that C_i+I_i is increasing in i, we can extend (10) as follows:

$$C_N+I_N>\cdots>C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}}>\cdots>C_j+\frac{I_j}{P_{1j}}. \quad (11)$$

(11) can be separated in the following way:

$$C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{jj}}, \quad \cdots \quad ,C_{j+1}+I_{j+1}>C_j+\frac{I_j}{P_{1j}},$$
$$C_N+I_N>C_j+\frac{I_j}{P_{jj}}, \quad \cdots \quad ,C_N+I_N>C_j+\frac{I_j}{P_{1j}}.$$

By Proposition 1, these inequalities imply that for any $Y\in S(j)$ $Y\succ Y-E_j$ $j=1,2,\cdots,N-1$. ■

By symmetry of the graph, we have the following corollary.

Corollary 2. *If $X_1\succ X_1+E_j$ for $j=1,2,\cdots,N-1$ and C_i+I_i is increasing in i, then $Y-E_j\succ Y$ for any $Y\in S(j)$ and $j=1,2,\cdots,N-1$.*

Proposition 3 is very powerful. For example, if $X_N\succ X_N-E_j$ for all $j=1,2,\cdots,N-1$, then the preference directions of all arcs in the graph (J,A) are determined by doing only $N-1$ comparisons between level N and $N-1$. Similar results hold for X_1. In such extreme cases, the optimal inspection plan can be found using preference relations by the second level in either direction.

4.2 Optimum Algorithm

Two common searching techniques are the Breadth-First-Search (BFS) and Depth-First-Search (DFS) [1]. In BFS, we examine all of the incident arcs to a node before moving to a new level; the operation is "fanning-out" from a node successively. In DFS, we move to a new level as soon as a new level is found and penetrate deeply into the graph.

If $X\succ X^+$ and $X\succ X^-$ for some X, X^+, and $X^-\in J$, then we call X an isolated node in the graph (J,A). In order to find the optimal inspection plan, we have to find all of the isolated nodes of (J,A) as fast as possible. Before pruning nodes and arcs in the graph, we have to confirm that at the end of the pruning at least one isolated node will remain, in which the optimal inspection plan exists. This means that we need to show that the graph does not have a cycle with the given preference directions. The following proposition will state the result confirming the existence of at least one isolated node after pruning.

Proposition 4. *After determining the preference directions on all arcs in (J, A), (J, A) is always acyclic.*

Proof. Suppose (J, A) is cyclic, then (J, A) has at least one cycle. Consider a cycle Z consisiting of $X_i \rightarrow X_{i+1} \rightarrow \cdots \rightarrow X_i$. By the definition of preference direction, $X_i \rightarrow X_{i+1}$ implies $TEC(X_i) > TEC(X_{i+1})$. Hence, the cycle Z implies $TEC(X_i) > TEC(X_{i+1}) > \cdots > TEC(X_i)$, which is a contradiction.

Therefore, (J, A) is acyclic. ■

The existence of an isolated node is confirmed by Proposition 4. We need to set the criteria for a path to follow on the graph. Consider two nodes X_1 and X_N, and their corresponding incident arcs, where X_i is defined as any node at level i on (J, A):

$$A_1 = (X_1, X_2) \text{ and } A_N = (X_N, X_{N-1}).$$

The preference relation of A_1 does not give any information about the other arcs. On the other hand, even though the preference relation of A_N fixes at most four[3] stages, there are 2^{N-4} other arcs[4] where the preference relation of A_N can be applied. It is better in terms of pruning speed to start a path from X_N than from X_1.

Rule 1: Use the deleting direction, $S_N \Rightarrow S_1$, in pruning the graph.

Rule 2: If the $TEC(X)$'s of two adjacent nodes are the same, select the node at the lower level and continue.

From now on, we will consider only deleting direction. Among the adjacent nodes at the next level to a given node, call the rightmost node in (J, A) the first node and the leftmost node the last node. Denote the first node at level i by X_i^f and the last node by X_i^l. For example, in five stage problem there are four adjacent nodes at level 4 to X_5. The First node X_4^f is (1,1,1,0, 1) and the last node X_4^l is (0,1,1,1,1). Consider $X_N = (1,1,\cdots,1)$ and its adjacent nodes at the level $N-1$: $X_{N-1}^f = (1,1,\cdots,1,0,1)$ and $X_{N-5}^l = (0,1,\cdots,1,1,1)$. Then the arc $A_N = (X_N, X_{N-1}^f)$ has three interrelated stages[5] by Theorem 1, so there are 2^{N-3} other arcs that have the same relation as A_N. Similarly, the same holds the case when the arc is (X_N, X_{N-1}^l). However, when any other

3) Among N stages we know that three or four stages' variables are fixed by Theorem 1.

4) With $N-4$ remained unfixed stages, there can be existed 2^{N-4} arcs together with the fixed four stages.

5) *i.e.*, stages $N-2$, $N-1$, and N.

plan at level $N-1$ is evaluated, the preference relation will only be applicable to 2^{N-1} other arcs. For example, consider the other node $X_{N-1}=(1,1,\cdots,1,0,1,1)$ at level $N-1$, in which four stages[6] are fixed with deleting the inspection station at stage $N-2$. Hence, there are only 2^{N-1} arcs which have the same preference relation as the arc (X_N, X_{N-1}). Therefore, selecting either the first or the last node is better in terms of the number of arcs where the preference relation can be applied. However, we are still unclear about which node should be selected as the next adjacent node. The following observation about the structure of the graph makes this ambiguity clear:

Observation 1. *The structure of graph (J, A) suggests that if we select the first node at each level, a path will only go through the right side of (J, A). On the other hand, if we continuously choose the last node, a path will go only through the left side of (J, A). Intuitively, a zig-zagging path seems to reduce the amount of uncovered region in (J, A).*

Using the property and Observation 1, we can set the second criterion for the path to follow.

Rule 3: When we select the adjacent node at the next level, follow the zig-zagging policy which takes the first and the last node alternatively at each level.

A detailed outline of the optimum algorithm is given as follows:

step 1. This is the highest level N of (J, A). Select the incident arc to the first node at the level $N-1$, $A_N=(X_N, X'_{N-1})$, and find the preference relation through Theorem 1. Mark the preference direction on A_N and cross out the dominated node, so that all incident arcs to the dominated node will not be considered any more. By the propositions, apply the preference relation of A_N to the other applicable arcs on (J, A). Mark the preference direction on these arcs and then cross out the corresponding dominated nodes too.

step 2. Establish the new starting node as X_{N-1}. Using the zig-zagging path policy, select the incident arc to the last node at level $N-2$, $A_{N-1}=(X'_{N-1}, X'_{N-2})$, and find its preference relation. Repeat the procedure in step 1.

step 3. Establish the next starting node as X'_{N-2}. With the same policy, select the arc $A_{N-2}=(X'_{N-2}, X'_{N-3})$ and find its preference relation.

step 4. Repeat the procedure in step 1 applying the zig-zagging path policy until level 1 of (J, A) is reached.

step 5. After pruning all of the dominated nodes and their incident arcs, establish the

6) *i.e.*, stages $N-3$, $N-2$, $N-1$, and N.

starting node as the first node at the highest level with the remaining undominated nodes and their incident arcs. Repeat the procedure in step 2 until the lowest level is reached.

step 6. Repeat step 5 until only isolated nodes remain. Calculate the total expected costs of the remaining isolated nodes and then let the optimal inspection plan X^* be the node with the minimum total expected cost.

4.3 A Heuristic Approach

In the previous sections, we have outlined an optimum algorithm — an algorithm that always gives an optimum solution to the problem. When we compare it with the enumerative method, we see that using a graph provides a more efficient solution. However, in the case with a large number of stages, the optimum algorithm is also difficult and time-consuming, making it somewhat impractical.

In the real world, most problems have to be solved by a given deadline. Thus people either use a commerical code which is not tailor-made for the problem, or they invent a heuristic approach which intuitively seems to be correct and which sometimes obtains optimum solutions. When the optimum is hard and expensive to achieve, near-optimum is usually considered good enough.

In pruning the graph, a greedy selection rule can be applied due to the special structure. The idea is that the node which produces the largest reduction in the total expected cost among the adjacent nodes at the next level has the highest probability of being optimal. Based on this conjecture, we describe the following heuristic algorithm for finding the optimal inspection plan:

step 1. Let the starting node be X_N at the highest level N on (J, A). Select the incident arc having the largest net gain among all of the incident arcs to X_N. If no incident arc has a net gain, then stop and X_N is optimal. Otherwise, follow the selected arc having the largest net gain to level $N-1$ and establish the corresponding node at level $N-1$ as a new starting node.

step 2. Repeat step 1 until there are no more incident arcs with a net gain.

5. Numerical Examples

The following examples illustrate how the algorithms developed in the previous section are used to find an optimal inspection plan. For the numerical examples, we use the five stage production line illustated in Figure 1. Two data sets for this problem are given in Table 1 and Table 2 to demonstrate the scheme of both algorithms and to show the computational results. Data set 2 is given to show the case where the Heuristic does not produce the optimal plan.

Table 1. Data set 1

	I_i	C_i	P_i
1	1	20	.03
2	2	40	.04
3	2	80	.03
4	2.5	140	.04
5	3	180	.05

Table 2. Data set 2

	I_i	C_i	P_i
1	2	20	.05
2	2	40	.64
3	1.5	50	.04
4	1.5	60	.06
5	2	80	.05

Two pages of Appendix show the sequential flow taken to find the isolated nodes. The first page is a result of Data set 1 and the second one is from Data set 2.

After identifying the isolated nodes, we have to compute their total expected costs:

Data set 1	Data set 2
$TEC(X=(1,0,1,0,1))=25.8964$	$TEC(X=(0,0,1,0,1))=17.6889$
$TEC(X=(1,0,0,1,1))=28.8319$	$TEC(X=(1,0,0,1,1))=17.9095$
$TEC(X=(0,1,1,0,1))=25.8668$	$TEC(X=(0,1,0,1,1))=17.6840$
$TEC(X=(0,1,0,1,1))=26.4549$	

The following two tables show the computational results of the Data sets.

Table 3. Computational results with Data set 1

	Enumerative	Optimum	Heuristic
the number of computations of total expected costs	16	4	1
the number of simple comparisons	0	5	9
the optimal inspection plan	(0,1,1,0,1)	(0,1,1,0,1)	(0,1,1,0,1)
the optimal cost	25.8668	26.8668	25.8668

Table 4. Computational results with Data set 2

	Enumerative	Optimum	Heuristic
the number of computations of total expected costs	16	3	1
the number of simple comparisons	0	8	10
the optimal inspection plan	(0,1,0,1,1)	(0,1,0,1,1)	(0,0,1,0,1)[7]
the optimal cost	17.6840	17.6840	17.6889

As shown in Tables 3 and 4, instead of 16 complicated computations of total expected costs in the enumerative method, both algorithms need only about half as many simple comparisons. Even though the heuristic approach may not always produce the exact optimal solution, the difference of the total costs between the optimum and the near-optimal plan found appears to be

7) In fact, this plan is not optimal. However, the heuristic finds it as optimal.

small enough to use the heuristic for relatively larger problems that require increasing efforts to find the optimal by the optimal algorithm.

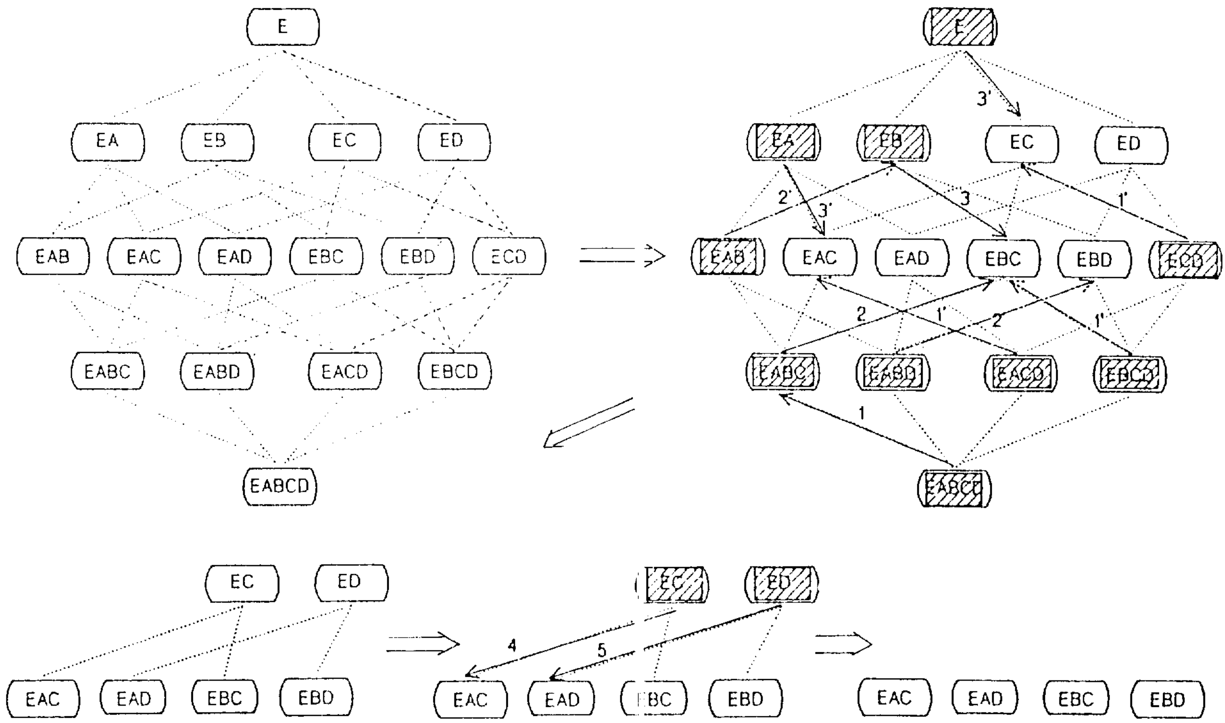
6. Conclusion

The algorithm developed in this paper provides the optimal inspection configuration for a flexible flow line. This is an important decision prior to production. The flexible flow line has been widely used in manufacturing systems recently. Even though the primary assumption of perfect inspection might not hold in most situations, the algorithm considered in this paper can be applied to a variety of problems.

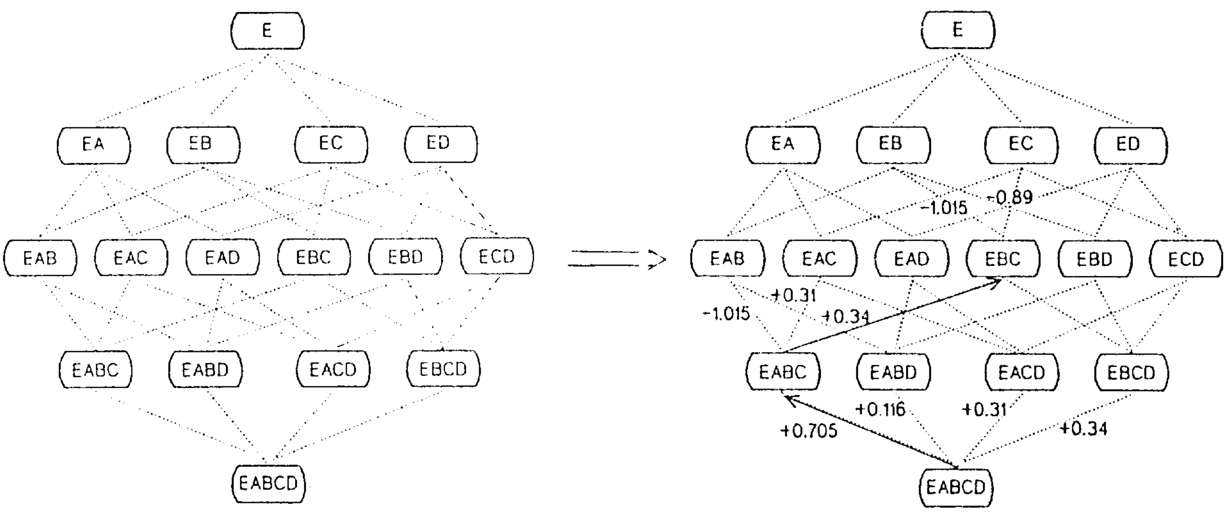
This paper has examined an inspection location problem for a serial type production line, which can be extended and generalized. Our future plans include of implementing the algorithms in some real situations. We also intend to extend this approach to the non-serial case in which some mixed flow of several parallel production lines exists. Finally, we hope to find properties of the model when the inspection is imperfect instead of perfect.

Appendix

optimum algorithm

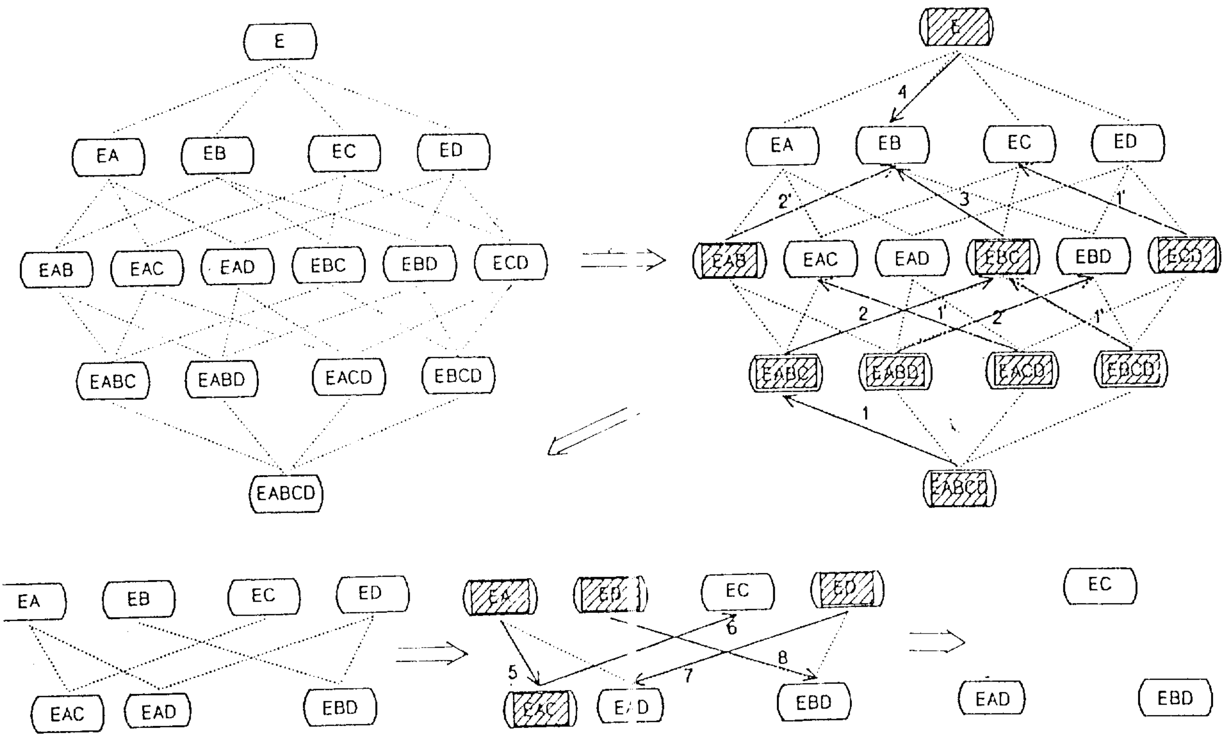


heuristic algorithm

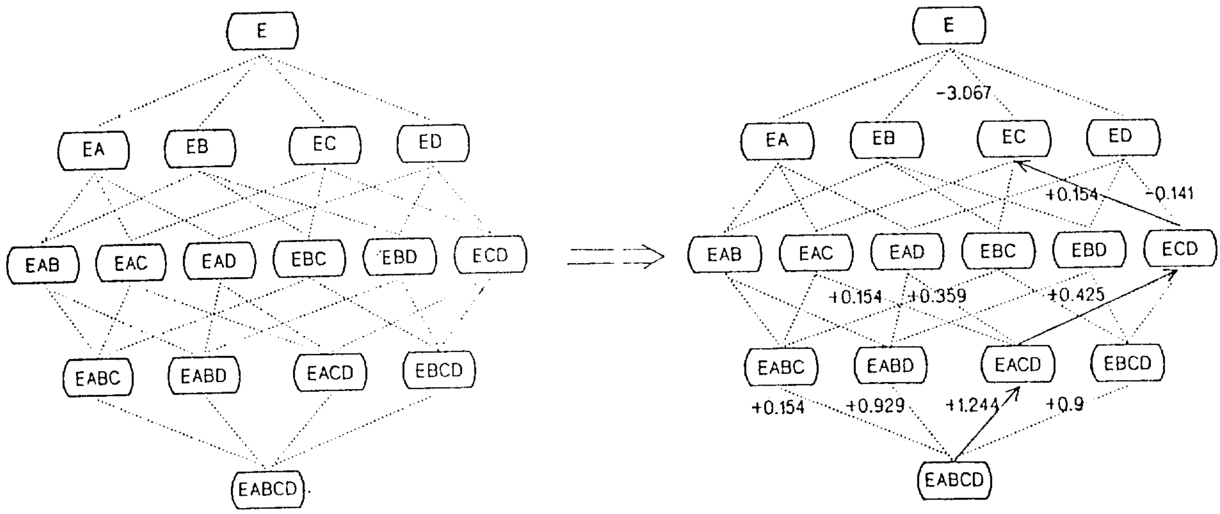


Appendix

optimum algorithm



heuristic algorithm



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