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MULTIPLICATIVE GROUP IN A FINITE RING

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1. Introduction and basic definitions

Let R be a finite ring with identity 1 and let G denote the multiplicative group of all units of R. An element e in R is said to be idempotent if $e^2 = e$. A nonzero idempotent is said to be primitive if it cannot be written as the sum of two orthogonal nonzero idempotents.

In [4], Artinian proved that if R is a semisimple Artinian ring, then R is isomorphic to a direct product of finite number of matrix rings over division rings. In particular, if R is finite, then we obtain the following;

THEOREM 1.1. *(Wedderburn-Artin's Structure Theorem for a finite ring)*
If R is a finite ring and J is the Jacobson radical of R, then $R/J \cong \oplus_{i=1}^{n} M_i$ where M_i is the ring of all $n_i \times n_i$ matrices over a finite field F_i.

In this paper, we will show that the multiplicative group G in a finite ring R with identity 1 has a (B, N)-pair satisfying the following conditions;

(1) $G = BNB$ where B and N are subgroups of G.
(2) $B \cap N$ is a normal subgroup of N and $W = N/(B \cap N)$, is generated by a set $S = \{s_1, s_2, \ldots, s_r\}$ where $s_i \in N/(B \cap N)$, ${s_i}^2 \equiv 1$ and $s_i \neq 1$.
(3) For any $s \in S$ and $w \in W$, we have $sBw \subset BwB \cup BswB$.
(4) We have $sBs \not\subseteq B$ for any $s \in S$.

When G, B, N and S satisfy the above conditions, we say that the quadruple (G, B, N, S) is a Tits system. The group W is called the Weyl gorup of the Tits system.

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2. Muultiplicative group in a matrix ring over a (finite) field

Let $R = M_n(F)$ be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let E be the set of all primitive idempotents in R and let $I(E)$ be the set of all rignt ideals in R which is generated by the subset S of E such that each member of S is orthogonal to another. Let $\triangle(E)$ be the set of all chains in $I(E)$. Let $\sum = \{e_1, e_2, \ldots, e_n\}$ be a subset of E satisfying that $e_ie_j = 0$ for $i \neq j$ and $e_1 + e_2 + \ldots + e_n = 1$.

Let $\sum_\triangle$ be the set of all chains in $\triangle(E)$ constructed by $\sum$. Pick a maximal chain c in $\sum_\triangle$;

$$c : e_1R + \ldots + e_{n-1}R \supset e_1R + \ldots + e_{n-2}R \supset \ \ldots \supset e_1R.$$

For each $g \in G$, define $g(C)$ by

$$g(e_1R + \ldots + e_{n-1})R \supset g(e_1R + \ldots + e_{n-2}R) \supset \ \ldots \supset g(e_1R).$$

Note that by the Proposition 3 in [6,pp 77], $g(C)$ is also a maximal chain in $\triangle(E)$ constructed by a set $\{f_1, f_2, \ldots, f_i\}$ satisfying that $f_if_j = 0$ for $i \neq j$ and $f_1 + f_2 + \ldots + f_n = 1$ where $f_i = ge_ig^{-1}$ for each $i = 1, 2, \ldots, n$.

LEMMA 2.1. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $B = \{g \in G : g(C) = C\}$. Then B is a subgroup of G and $B = \{g \in G : g$ is upper-triangular$\}$.*

Proof. Let $g = (g_{ij}) \in B$. Without loss of generality, we may take $e_i = E_{ij} = (a_{ij})$ where $a_{ij} = 1$ and $a_{ij} = 0$ for $i \neq j (1 \leq i, j \leq n)$, which is assured by the Proposition 3 in [6,p 77]. From the equality $g(C) = C$, we have

$$g(e_1R) = e_1R - (1)$$

$$g(e_1R + e_2R) = e_1R - (2)$$

$$\cdots\cdots\cdots$$

$$g(e_1R + \ldots + e_{n-1})R = e_1R + \ldots + e_{n-2}R - (n-1).$$

From the equality (1), $g_{1j} = 0$ for $2 \leq j \leq n$. From the equality (2), $g_{2j} = 0$ for $3 \leq j \leq n$. By induction on n and the equality ($n-1$), $g_{ij} = 0$ for $i \leq j \leq n$. Hence $g = (g_{ij})$ is upper-triangular. On the other hand, it is clear that $g(C) = C$ for all upper-triangular $g \in G$. Consequently, $B = \{g \in G : g \text{ is upper-triangular}\}$ and clearly, B is a subgroup of G.

LEMMA 2.2. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $N = \{g \in G : g(\sum_\triangle) = \sum_\triangle\}$. Then N is a subgroup of G and $N = \{g \in G : g$ has only one nonzero entry for each row$\}$.*

Proof. Clearly, N is a subfroup of G. Let $g = (g_{ij}) \in N$. Without loss of generality, we may assume that the first row of g has two nonzero entries $g_{1r}, g_{1s} (1 \leq r < s \leq n)$. Let C be a maximal chain in $\sum_\triangle$;

$$C : e_1R + \ldots + e_{r-1}R + e_{r+1}R + \ldots + e_sR + \ldots + e_nR \supset \ldots \supset e_1R.$$

Then we have $g(e_1R + \ldots + e_{r-1}R + e_{r+1}R + \ldots + e_sR + \ldots + e_nR) = R$, which means that $g(C)$ is not a maximal chain in $\sum_\triangle$, a contraction.

On the other hand, it is clear that if $g \in G$ has only one nonzero entry for each row, then $g(\sum_\triangle) = \sum_\triangle$. Hence we have the result.

LEMMA 2.3. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $N_G(D) = \{g \in G : gDg^{-1} \subseteq D\}$ where $D = \{a \in R : a$ is diagonal matrix$\}$. Then $N_G(D) = N$ where N is given in Lemma 2.2.*

Proof. First, we will show that $N \subseteq N_G(D)$. Indeed, For any $g = (g_{ij}) \in N, d = (d_{ij}\delta_{ij}) \in D$ and $g^{-1} = (h_{ij})$, we have

$$gdg^{-1} = \sum_{k=1}^{n} (g_{ik}h_{kj})d_{kk} = ((g_{is}h_{sj})d_{ss}) \text{ (for some } s, 1 \leq s \leq n)$$
$$= \sum_{k=1}^{n} (g_{ik}h_{kj})d_{ss} = (d_{ss}\delta_{ij}) \in D.$$

Hence $g \in N_G(D)$. Assume that $N \neq N_G(D)$. Then there exists $g = (g_{ij}) \in N_G(D) \backslash N$. Without loss of generality, we may assume that some row of g, say i-th row, has two non-zero entries $g_{ir}, g_{is}(1 \leq r < s \leq n)$. Choose a diagonal matrix $d = (d_{ij}\delta_{ij})$ so that $d_{ii} = 0$ for $i \neq r$, s and $d_{rr}, d_{ss} \neq 0$ and $d_{rr} \neq d_{ss}$. Since $g \in N_G(D)$, there exists $d_1 = (x_{ij}\delta_{ij}) \in D$ such that $gd = d_{1g}$ – $(*)$. Note that (i,r)-entry of $gd = g_{ir}d_{rr}$, (i,s)-entry of $gd = g_{is}d_{ss}$, (i,r)-entry of $d_{1g} = x_{ii}g_{rr}$ and (i,s)-entry of $d_{1g} = x_{ii}g_{ss}$. From $(*)$, we have $d_{rr} = x_{ii}$ and $d_{ss} = x_{ii}$, and so $d_{rr} = d_{ss}$, a contraction. Hence $N = N_G(D)$.

PROPOSITION 2.4. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $B = \{g \in G : g(C) = C\}$ and $N = \{g \in G : g(\sum_{triangle}) = \sum_{\triangle}\}$. Then $G = BNB$.*

Proof. Let g be any element in G. To make each row of G have different number of zeroes beginning from the left, consider an invertible matrix obtained from g by means of a sequence of elemently row operation which is defined by replacing i-th row with i-th row $+a \cdot (j$-th row) for some nonzero $a \in F(i < j)$. Note that the matrices obtained from identity matrix by the operations given above are upper-triangular, and so $bg = g_1$ for some $b \in B$ by Lemma 2.1.

Next, we can make an invertible matrix obtained from g_1 by means of a sequence of some rotation of two rows be upper-triangular. Note that the matrices obtained from identity matrix by the operations given above are contained in N by Lemma 2.2. Hence $ng_1 = nbg = b_1$ for some $b_1 \in B$, so $g = (nb)^{-1}b_1 = b^{-1}n^{-1}b_1 \in BNB$.

PROPOSITION 2.5. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $B = \{g \in G : g(C) = C\}$ and $N = \{g \in G : g(\sum_{\triangle}) = \sum_{\triangle}\}$. Then $B \cap N$ is a normal subgroup of G.*

Proof. It is clear from Lemma 2.1 and Lemma 2.2 to show that $B \cap N = G \cap D$. For all $n \in N$ and all $a \in B \cap N, nan^{-1} \in D$ since $a \in D$ and $n \in N$. Clearly, $nan^{-1} \in G$. Hence $nan^{-1} \in G \cap D = B \cap N$, and so $B \cap N$ is a normal subgroup of G.

PROPOSITION 2.6. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $B = \{g \in G : g(C) = C\}$ and $N = \{g \in G : g(\sum_{\triangle}) = \sum_{\triangle}\}$. Then $N/(B \cap N)$*

is generated by a set $S = \{s_1, s_2, \ldots, s_k\}$ *for some positive integer* k *where* $s_i \in N/(B \cap N), s_i{}^2 \equiv 1$ *and* $s_i \neq 1$.

Proof. Let t_{ij} be the matrix obtained by interchanging two defferent rows, i-th and j-th rows, on the identity matrix I. Then $t_{ij}^2 = I$ and $t_{ij} \neq I$. Let $s_{ij} = t_{ij}(B \cap N)$. Since $t_{ij} \in N/(B \cap N)$.

We will show that $N/(B \cap N)$ is generated by the set $\{s_{ij} | 1 \leq i, j \leq n, i \neq j\}$. Let $n(B \cap N) \in N/(B \cap N)$ and $t = (d_{ij}\delta_{ij}) \in B \cap N$ where d_{ii} is the inverse of nonzero entry of i-th column of n. Then nt can be expressed by a product of some elements of $\{t_{ij} | 1 \leq i, j \leq n, i \neq j\}$. Therefore, $nt(B \cap N) = (n(B \cap N))(t(B \cap N)$ can by also expressed by a product of some element of $\{s_{ij} | 1 \leq i, j \leq n, i \neq j\}$.

COROLLARY 2.7. *Let* R *be the ring of all* $n \times n$ *matrices over a (finite) field* F *and let* G *the multiplicative group of* R. *Let* $B = \{g \in G : g(C) = C\}$ *and* $N = \{g \in G : g(\sum_\triangle) = \sum_\triangle\}$. *Then* $N/(B \cap N)$ *is generated by a set* $S_0 = \{s_{ii+1} | \; i = 1, 2, \ldots, n-1\}$ *where* $s_{ii+1} = t_{ii+1}(B \cap N) \in N/(B \cap N)$ *and* t_{ii+1} *be the matrix obtained by interchanging two rows,* i*-th and* $i+1$*-th rows, on the identity matrix* I.

Proof. Since the symmetric group S_n of degree n is generated by all the $n-1$ transposition $(i\, i+1)$ for $i = 1, 2, \ldots, n-1$, each s_i in the set S given in Proposition 2.6 is gerarated by S_0.

DEFINITION 2.8. An element $s_{ij} \in W = N/(B \cap N)$ satisfying $s_{ij}^2 \equiv 1$ and $s_{ij} \not\equiv 1$ (1 is an identity of W) is called the involution with respect to B. The group W generated by the involutions is called the Weyl group of the $\sum_\triangle$.

The equality $G = BNB$ which is proved in proposition 2.4 shows that G is a union of the double cosets with respect to (B, B) and that we can take an element of N as representative from each (B, B)-double coset. For $w = n(B \cap N) \in W$, we can define $BwB = BnB$. We also use notations such as $Bw = Bn$ when $w = n(B \cap N)$.

PROPOSITION 2.9. *Let* R *be the ring of all* $n \times n$ *matrices over a (finite) field* F *and let* G *the multiplicative group of* R. *Let* $B = \{g \in G : g(C) = C\}$ *and* $N = \{g \in G : g(\sum_\triangle) = \sum_\triangle\}$. *Then* $s_{ij}Bs_{ij} \subseteq B$

for any $s_{ij} \in S$, a generating set for $W = N/(B \cap N)$ given in the proof of proposition 2.6.

Proof. Assume that there exists some $s_{ij} \in S$ such that $s_{ij}Bs_{ij} \subseteq B$. Let $s_{ij} = t_{ij}(B \cap N)$ where t_{ij} is the matrix obtained by interchanging two rows, i-th and j-th rows $(i < j)$, on the identity natrix I. Choose $b \in B$ so that (i,j)-entry of $b = b_{ij} \neq 0$. Then $t_{ij}bt_{ij}^{-1} \notin B$. Hence $s_{ij}Bs_{ij}^{-1} = s_{ij}Bs_{ij} \not\subseteq B$ for any $s_{ij} \in S$.

PROPOSITION 2.10. *Let R be the ring of all $n \times n$ matrices over a (finite) field F and let G the multiplicative group of R. Let $B = \{g \in G : g(C) = C\}$ and $N = \{g \in G : g(\sum_\triangle) = \sum_\triangle\}$. Then $sBw \subset BsB \cup BswB$ for any $s \in S$, a generating set for W, and for any $w \in W$.*

Proof. Since W is generated by S, $w = s_1s_2 \ldots s_k$ for some $s_i \in S$ and positive integer $k(i = 1, 2, \ldots, k)$. We will prove the result by induction on k. By straight forward calculation, $sBs_1 \subset BsB$ if $s = s_1$ and $sBs_1 \subset Bss_1B$ if $s \neq s_1$. Moreover, $sBs_1s_2 \subset BsB$ if $s = s_1 = s_2$ and $sBs_1s_2 \subset Bss_1s_2B$ otherwise. Hence by induction on k, $sBw = sBs_1s_2 \ldots s_k \subset BsB$ if $s = s_i$ for all $i = 1, 2, \ldots, k$ and $sBw \subset BswB$ otherwise.

DEFINITION 2.11. Let G be a group. Two subgroups B and N of G are said to be a (B, N)-pair of G if the following conditions are satisfied:

(1) $G = BNB$ where B and N are subgroups of G.
(2) $B \cap N$ is a normal subgroup of N and $W = N/(B \cap N)$ is generated by the set $S = \{s_1, s_2, \ldots, s_k\}$ where $s_i \in W$, $s_i^2 \equiv 1$ and $s_i \not\equiv 1$ (1 is the identity of W).
(3) For any $s \in S$ and $w \in W$, we have $sBw \subset BwB \cup BswB$.
(4) For any $s \in S$, we have $sBs \not\subseteq B$.

When G, B, N, and S satisfy the above conditions, we say that the quadruple (G, B, N, S) is a Tits system. The group W is called the Weyl group of the Tits system.

In this section, we have shown that if R is the ring of all $n \times n$ matrices over a (finite) field F and G is the multiplicative group of R, then there is a Tits system (G, B, N, S).

COROLLARY 2.12. *If G is the multiplicative group of a finite semisimple ring R, then there is a Tits system $(G, B\ N, S)$.*

Proof. By Theorem 1.1, $R \cong \oplus_{i=1}^{n} M_i$ where M_i is the matrix ring of all $n_i \times n_i$ matrices over a finite field F_i. For the simplicity of notation, we can assume that $R = \oplus_{i=1}^{n} M_i$. Let G_i be the multiplicative group of M_i for each $i = 1, 2, \ldots, n$. By the above argument, there is a Tits system (G_i, B_i, N_i, S_i) for each $i = 1, 2, \ldots, n$.

Note that $G = \oplus_{i=1}^{n} G_i$. Let $B = \oplus_{i=1}^{n} B_i$, $N = \oplus_{i=1}^{n} N_i$, and $S = \oplus_{i=1}^{n} S_i$. It is easy to show that (G, B, N, S) is a Tits system for G.

3. Multiplicative Group in a Finite Ring with Identity

In this section, we will show that in the multiplicative group G of a finite ring with identity 1, there is a Tits system (G, B, N, S).

LEMMA 3.1. *Let R be a finite ring with identity 1, let J the Jacobson radical of R, let G the multiplicative group of R and let $\overline{G}$ the multiplicative group of $\overline{R} = R/J$. Then $g \in G$ if and only if $g + J \in \overline{G}$.*

Proof. $(\Rightarrow)$ Clear.

$(\Leftarrow)$ Suppose that $\overline{g} = g + J \in \overline{G}$. Then there exists $\overline{h} = h + J$ such that $\overline{g}\overline{h} = \overline{h}\overline{g} = \overline{1}$ (where $\overline{1}$ is the identity of $\overline{G}$), and so $1 - gh$ and $1 - hg \in J$. Since J is a two-sided quasi left ideal of R and R has identity $1, 1-(1-gh) = gh$ and $1-(1-hg) = hg$ are invertible in G by Theorem 2.3 through Lemma 2.8, in [4, pp.426-428]. Hence $(gh)x = y(hg) = 1$ for some x and $y \in G$. Therefore, $g \in G$.

LEMMA 3.2. *Let $\phi : A \to B$ be a ring homomorphism which is onto. If P and Q are subsets of B, then $\phi^{-1}(PQ) = \phi^{-1}(P)\phi^{-1}(Q)$.*

Proof. If $a \in \phi^{-1}(PQ)$, then $\phi(a) \in PQ$, $\phi(a) = pq$ for some $p \in P$ and $q \in Q$. Since ϕ is onto, there exist p_0 and $q_0 \in A$ such that $\phi(p_0) = p$ and $\phi(q_0) = q$. Then $\phi(a) = \phi(p_0)\phi(q_0) \in \phi(\phi^{-1}(P)\phi^{-1}(Q))$, and so $a \in \phi^{-1}(P)\phi^{-1}(Q)$.

If $a \in \phi^{-1}(P)\phi^{-1}(Q)$, then $a = xy$ for some $x \in \phi^{-1}(P)$ and $y \in \phi^{-1}(Q)$. Thus $\phi(a) = \phi(xy) = \phi(x)\phi(y) \in PQ$, and so $a \in \phi^{-1}(PQ)$.

PROPOSITION 3.3. *Let R be a finite ring with identity and G be the multiplicative group of R. Then G has a (B^*, N^*)-pair satisfying the following conditions;*

(1) *$G = B^*N^*B^*$ where B^* and N^* are subgroups of G.*
(2) *$B^* \cap N^*$ is a normal subgroup of N^* and $W^* = N^*/(B^* \cap N^*)$ is generated by a set $S^* = \{s_1^*, s_2^*, \ldots, s_k^*\}$ where $s_i \in N^*/(B^* \cap N^*)$, $s_i^{*2} \equiv 1$ and $s_i^* \not\equiv 1$.*
(3) *For any $s^* \in S^*$ and $w^* \in W^*$, we have $s^*B^*w^* \subset B^*w^*B^* \cup B^*s^*w^*B^*$.*
(4) *We have $s^*B^*s^* \not\subseteq B^*$ for any $s^* \in S^*$.*

Proof. Let $\pi : R \to \overline{R} = R/J$ be the canonical ring homomorphism where J is the Jacobson radical of R. For any $g \in G$, $\pi(g) = g + J = \overline{g} \in \pi(G) = \overline{G}$ by Lemma 3.1. By Corollary 2.12, $\bar{G} = BNB$ for some subgroups B and N of $\bar{G}$. Thus it follows from Lemma 3.2 that $g \in \pi^{-1}(\bar{G}) = \pi^{-1}(BNB) = \pi^{-1}(B)\pi^{-1}(N)\pi^{-1}(B)$. Hence $G = \pi^{-1}(B)\pi^{-1}(N)\pi^{-1}(B)$. Clearly, $B^* = \pi^{-1}(B)$ and $N^* = \pi^{-1}(N)$ are subgroups of G. If $n \in N^*$ and $a \in B^* \cap N^*$, then $\pi(n) \in N$ and $\pi(a) \in B \cap N$. Note that $\pi(n)\pi(a)\pi(n)^{-1} = \pi(nan^{-1}) \in B \cap N$ since $B \cap N$ is a normal subgroup of N. Thus $nan^{-1} \in \pi^{-1}(B \cap N) = \pi^{-1}(B) \cap \pi^{-1}(N) = B^* \cap N^*$, and so $B^* \cap N^*$ is a normal subgroup of N^*.

Let $n_0(B^* \cap N^*)$ be arbitrary element of $N^*/(B^* \cap N^*)$. By Corollary 2.12, $N/(B \cap N)$ is generated by a subset $S = \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_k\}$. of $N/(B \cap N)$ where $s_i^2 \equiv 1$ and $s_i \not\equiv 1$. Let $n = \pi(n_0) \in N$. Then $n(B \cap N)$ can be a finite product of elements of S, say $n(B \cap N) = \overline{s}_1 \cdot \overline{s}_2 \ldots \overline{s}_t = s_1 \cdot s_2 \ldots s_t(B \cap N)$ where $\overline{s}_i = s_i(B \cap N)$, $i = 1, 2, \ldots, t$ for some positive integer t. Note that $n(s_1 \cdot s_2 \ldots s_t)^{-1} \in B \cap N$. Since π is onto, there exists $s_i^* \in N^*$ such that $\pi(s_i^*) = s_i$ for each $i = 1, 2 \ldots t$. Thus $n(s_1 \cdot s_2 \ldots s_t)^{-1} = \pi(n_0)\pi(s_1^* \cdot s_2^* \ldots s_t^*)^{-1} = \pi(n_0(s_1^* \ldots s_t^*)^{-1}) \in (B \cap N$, and hence $n_0(s_1^* \cdot s_2^* \ldots s_t^*)^{-1} = \pi^{-1}(B \cap N) = B^* \cap N^*$, $n_0(B^* \cap N^*) = (s_1^* \cdot s_2^* \ldots s_t^*)(B^* \cap N^*) = s_1^*(B^* \cap N^*) \cdot s_2^*(B^* \cap N^*) \ldots s_t^*(B^* \cap N^*)$, which means that $n_0(B^* \cap N^*)$ can be a finite product of elements of $S = \{\overline{s}_1^*, \overline{s}_2^*, \ldots, \overline{s}_k^*\}$. Therefore, $N^*/(B^* \cap N^*)$ is generated by a subset $S^* = \{s_1^*, s_2^*, \ldots, s_k^*\}$.

Since $\overline{s}B\overline{s} \not\subseteq B$ for any $\overline{s} \in S$, it is easy to show that $\overline{s}^*B\overline{s}^* \not\subseteq B^*$

for any $\overline{s} \in S^*$. It remains to show that $\overline{s}^* B\overline{w}^* \subset B^*\overline{w}^*B^* \cup B^*\overline{s}^*\overline{w}^*B^*$ for any $\overline{s}^* \in S^*$ and $\overline{w}^* \in W^*$. Let $\pi(\overline{s}^*) = \overline{s}$ and $\pi(\overline{w}^*) = \overline{w}$ for any $\overline{s}^* \in S^*$ and $\overline{w}^* \in W^*$. Then $\overline{s} \in S$ and $\overline{w} \in W$, and so by Corollary 2.12, we have $\overline{s}B\overline{w} \subset B\overline{w}B \cup B\overline{s}\,\overline{w}B$. Hence it follows from Lemma 3.2 that $\overline{s}^*B^*\overline{w}^* \subset \pi^{-1}(\overline{s})\pi^{-1}(B)\pi^{-1}(\overline{w}) = \pi^{-1}(\overline{s}B\overline{w}) \subset \pi^{-1}(B\overline{s}B \cup B\overline{s}\,\overline{w}B) \subset \pi^{-1}(B\overline{s}B) \cup \pi^{-1}(B\overline{s}\,\overline{w}B) = \pi^{-1}(B)\pi^{-1}(\overline{s})\pi^{-1}(B) \cup \pi^{-1}(B)\pi^{-1}(\overline{s})\pi^{-1}(\overline{w})\pi^{-1}(B) = B^*\overline{w}^*B^* \cup B^*\overline{s}^*B^* \cup B^*\overline{s}^*\overline{w}^*B^*$.

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