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ASYMPTOTICS OF A CLASS OF ITERATED RANDOM MAPS

CHANHO LEE

1. Introduction

Let (S,ρ) be a metric space, Γ a set of measurable maps on S into itself, $\Im$ a σ-field on Γ such that the map $(\gamma, x) \to \gamma(x)$ is measurable on $(\Gamma \times S, \Im \otimes \mathcal{B}(S))$ into $(S, \mathcal{B}(S))$. Let $\mathbf{P}$ be a probability measure on $(\Gamma, \Im)$. On some probability space $(\Omega, \mathcal{F}, Q)$ define a sequence of i.i.d. random maps $\alpha_1, \alpha_2, \ldots$ with common distribution $\mathbf{P}$. For a given random variable X_0, independent of the sequence α_n, define $X_1 = \alpha_1 X_0, \cdots, X_n = \alpha_n X_{n-1} = \alpha_n \cdots \alpha_1 X_0$. Then X_n is a Markov process with transition probability $p(x, dy)$ given by

$$p(x,B) = \mathbf{P}(\{\gamma \in \Gamma; \gamma(x) \in B\}),\ x \in S,\ B \in \mathcal{B}(S). \tag{1}$$

We often write $X_n(x)$ for X_n in case $X_0 = x$. Denote by $\mathbf{P}^n$ the joint distribution of $\alpha_1, \cdots, \alpha_n$, i.e., $\mathbf{P}^n = \mathbf{P} \times \cdots \times \mathbf{P}$ on $(\Gamma^n, \Im^{\otimes n})$

A Probability measure π on $(S, \mathcal{B}(S))$ is said to be *invariant* for p if $\pi(B) = \int p(x,B)\pi(dx),\ B \in \mathcal{B}(S)$. We shall write $p^{(n)}(x,dy)$for the n-step transition probability with $p^{(1)} = p$. Then $p^{(n)}(x,dy)$ is the distribution of $\alpha_n \cdots \alpha_1 x$.

In this article S is a topologically complete subspace of $\mathbf{R}^1$ i.e., the relativized topology on S may be metrized so as to make S complete. $\mathcal{B}(S)$ is the Borel σ-field of S.

For Γ one takes a set of measurable monotone (increasing or dereasing) functions on S into itself.

Make the assumption on $\mathbf{P}$:

There exists x_0 and a positive integer n_0 such that

$$Q(X_{n_0}(x) \le x_0 \quad \forall x) > 0, \quad Q(X_{n_0}(x) \ge x_0 \quad \forall x) > 0.$$

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This defines a topology on $\mathcal{P}(S)$ that is stronger than the weak-star topology.

Our main result is

THEOREM 2.1. *Suppose there exists a positive n_0 and some $x_0 \in S$ such that (2) holds. Then there exists a unique invariant probability π for $p(x,dy)$ (and $p^{(n)}(x,dy)$ converges exponentially fast to π in the $d-$metric, uniformly in x).*

First we state a lemma proved by Bhattacharya(1988) as follows.

LEMMA 2.2. *The space $\mathcal{P}(S)$ is complete under the distance d defined by (3).*

Proof. It is known that $\mathcal{P}(S)$ is topologically complete under the weak-star topology [Parthasarthy (1967),p.26], which is weaker than its topology under d. Hence if μ_n is a sequence in $\mathcal{P}(S)$ such that $d(\mu_n, \mu_m) \to 0$ as $n, m \to \infty$, then there exists $\mu \in \mathcal{P}(S)$ such that u_n converges weak-star to μ. Fix a continuous monotone nondecreasing γ on S into S and write F_n and F for the cumulative distribution functions of $\mu_n \circ \gamma^{-1}$ and $\mu \circ \gamma^{-1}$, respectively. Then $F_n(x)$ converges to $F(x)$ at all points of continuity of F. On the other hand, $\sup\{|F_n(x) - F_m(x)| : x \in \mathbf{R}^1\} \leq d(\mu_n, \mu_m)$. Hence F_n converges uniformly to a function that is necessarily right continuous. This implies that this limit function is F and that $F_n(x)$ converges to $F(x)$ uniform for all x. This being true for every continuous nondecreasing $\gamma, \mu_n(A)$ converges to $\mu(A)$ for every $A \in \mathcal{A}$. But μ_n converges uniformly on $\mathcal{A}$. Hence $d(\mu_n, \mu) \to 0$.

We notice that the lemma still works for the case of continuous monotone (increasing or decreasing) maps.

Proof of the theorem 2.1. First of all $\gamma^{-1}\mathcal{A} \subset \mathcal{A}$ $\forall \gamma \in \Gamma$, outside a set of zero $\mathbf{P}$-probability, and thus, by (3),

$$d(\mu \circ \gamma^{-1}, \nu \circ \gamma^{-1}) = \sup_{A \in \mathcal{A}} |\mu(\gamma^{-1}A) - \nu(\gamma^{-1}A)| \leq d(\mu, \nu),$$

i.e., $\mu \to \mu \circ \gamma^{-1}$ is a contraction. Define the (adjoint) operator T^* acting on $\mathcal{P}(S)$ by

$$(T^*\mu)(B) = \int p(x,B)\mu(dx).$$

Therefore, for $A \in \mathcal{A}_1 \cup \mathcal{A}_4$,

(5)

$$
\begin{aligned}
&|T^{*n_0}\mu(A) - T^{*n_0}\nu(A)| \\
&\quad \le \int_{\Gamma^{n_0}} |\mu((\gamma_{n_0}\cdots\gamma_1)^{-1}A) - \nu((\gamma_{n_0}\cdots\gamma_1)^{-1}A)|\mathbf{P}^{n_0}(d\gamma_1\cdots d\gamma_{n_0}) \\
&\quad \le \int_{\Gamma_1} |\mu((\gamma_{n_0}\cdots\gamma_1)^{-1}A) - \nu((\gamma_{n_0}\cdots\gamma_1)^{-1}A)|P^{n_0}(d\gamma_1\cdots d\gamma_{n_0}) \\
&\qquad + \int_{\Gamma^{n_0}\setminus\Gamma_1} |\mu((\gamma_{n_0}\cdots\gamma_1)^{-1}A) \\
&\qquad - \nu((\gamma_{n_0}\cdots\gamma_1)^{-1}A)|P^{n_0}(d\gamma_1\cdots d\gamma_{n_0}) \\
&\quad = \int_{\Gamma^{n_0}\setminus\Gamma_1} |\mu((\gamma_{n_0}\cdots\gamma_1)^{-1}A) \\
&\qquad - \nu((\gamma_{n_0}\cdots\gamma_1)^{-1}A)|P^{n_0}(d\gamma_1\cdots d\gamma_{n_0}) \\
&\quad \le (1 - \mathbf{P}^{n_0}(\Gamma_1)) \cdot d(\mu,\nu).
\end{aligned}
$$

For $A \in \mathcal{A}_2 \cup \mathcal{A}_3$, similarly,

(6)

$$
\begin{aligned}
&|T^{*n_0}\mu(A) - T^{*n_0}\nu(A)| \\
&\quad \le \int_{\Gamma^{n_0}} |\mu((\gamma_{n_0}\cdots\gamma_1)^{-1}A) - \nu((\gamma_{n_0}\cdots\gamma_1)^{-1}A)|P^{n_0}(d\gamma_1\cdots d\gamma_{n_0}) \\
&\quad \le (1 - P^{n_0}(\Gamma_2))d(\mu,\nu).
\end{aligned}
$$

Combining (5), (6) one gets

$$
(7) \qquad d(T^{*n_0}\mu, T^{*n_0}\nu) \le \max\{1 - P^{n_0}(\Gamma_1),\ 1 - P^{n_0}(\Gamma_2)\} \cdot d(\mu,\nu).
$$

(4) and (7) together imply

$$
d(T^{*n}\mu, T^{*n}\nu) \le \delta^{[n/n_0]} d(\mu,\nu), \quad \forall n = 1, 2, \cdots
$$

where $[n/n_0]$ is the integer part of n/n_0, and

$$
\delta = \max\{1 - P^{n_0}(\Gamma_1),\ 1 - P^{n_0}(\Gamma_2)\}.
$$

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DEPARTMENT OF MATHEMATICS, HAN NAM UNIVERSITY, TAEJON 300-791, KOREA