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CENTRAL SEPARABLE ALGEBRAS OVER LOCAL–GLOBAL RINGS I

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In [4], DeMeyer and Ingraham posed the following Question, and they show that (i) and (ii) are equivalent where R is conected.

QUESTION. For which connected commutative rings R are the followings true?

(i) If A is a central separable R-algebra, then every indecomposable finitely generated projective A-module is of the form $P \otimes_R E$ for E a fixed indecomposable finitely generated projective A-module and P in the projective class group of R.

(ii) If A is a central separable R-algebra, then there is a unique (up to isomorphism) central separable R-algebra D equivalent to A in $Br(R)$ so that D is connected and A° is isomorphic to $End_D(P, P)$ for some D-progenerator P.

The Question holds if R is a semilocal ring or a ring of polynomials in one variable over a perfect field [3]. Bass [2] has pointed out the existance of principal ideal domains where (i) and (ii) in the question fail.

In this paper, we show that if R is a local-global domain then the Question holds.

McDonald and Waterhouse in [6] and Estes and Guralnick in [5] introduced the concept of local-global rings (so called rings with many units) independently. A local-global ring is a commutative ring R with 1 satisfying; if a polynomial f in $R[x_1, \cdots, x_n]$ represents a unit over R_P for every maximal ideal P in R, then f represents a unit over R. Such rings include semilocal rings, or more generally, rings which are von Neumann regular mod their Jacobson radical, and the ring of all algebraic integers.

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All domains in this paper are commutative integral domains with 1, all modules are unitary right modules. When P is a module over a commutative ring R, we denote $P \otimes_R R_M$ by P_M where M is a maximal ideal of R.

We begin with two Lemmas.

LEMMA 1. *Let R be a domain, and an R-algebra A a projective R-module. If B is a projective A-module, then B_M is a projective A_M-module for each maximal ideal M of R.*

Proof. Since B is a projective A-module, there exists an A-module C and exists an A-isomorphism $g : B \oplus C \longrightarrow \oplus_i A_i$ where A_i is isomorphic to A for each i. Define $h : B_M \oplus C_M \longrightarrow \oplus_i (A_i)_M$ by $h(d/s) = g(d)/s$ where d is in $B \oplus C$ and s is in $R - M$. Then h is injective, since g is injective and since A is a torsion-free R-module. Hence h is an A_M-isomorphism.

LEMMA 2. *Let R be a domain, and an R-algebra A a finitely generated projective R-module. If B is an indecomposable finitely generated projective A-module, then B_M is an indecomposable finitely generated projective A_M-module for each maximal ideal M of R.*

Proof. Let $C_1 \oplus C_2$ be a nontrivial decomposition of B_M as A_M-modules. Set $E_i = \{b \in B | b/1 \in C_i\}$ for $i = 1, 2$. Then E_1 and E_2 are nontrivial A-modules clearly. Since B is finitely generated projective R-module, B is a torsion-free R-module. Therefore there is a natural imbedding g of B into B_M as R-modules, and E_i is the preimage of C_i by g for $i = 1, 2$. Thus we have $E_1 \cap E_2 = \{0\}$ and $B = E_1 + E_2$, *i.e.*, $E_1 \oplus E_2$ is a nontrivial decomposition of B as A-modules. Hence, by Lemma 1, we have the conclusion.

Now we are ready to prove Theorems.

THEOREM 1. *Let R be a local-global domain. If A is a central separable R-algebra, then any two indecomposable finitely generated projective A-modules are isomorphic.*

Proof. Let B and C be two indecomposable finitely generated projective A-modules, and M any maximal ideal of R. Then B_M and C_M

are indecomposable finitely generated projective A_M-modules by Theorem 2.1 of [1] and Lemma 2. But, since A_M is a central separable algebra over the local ring R_M by Corollary 1.6 of [1], B_M and C_M are isomorphic for all maximal ideal M of R by Theorem 1 of [3]. Hence B and C are isomorphic by Theorem 2.6 of [5].

THEOREM 2. *If R is a local-global domain, then (i) and (ii) in the Question hold.*

Proof. Theorem of [6] and Theorem 1.

And, similarly with Corollary 1 of [3], we can describe central separable algebras over a local-global domain as full matrix rings.

THEOREM 3. *Let A be a central separable algebra over a local-global domain R. Then A is isomorphic to $M_n(D)$ for a uniquely determined n and D which is described in the Question.*

Proof. By Theorem 2, there is a unique central separable R-algebra D equivalent to A in $Br(R)$ such that D is connected and $A^\circ \cong End_D(P,P)$ for some D-progenerator P. Since P is a finitely generated D-module, P is a direct sum of a finite number of indecomposable D-modules. But D is an indecomposable D-modules, since D is connected. Thus P is a finite direct sum of D by Theorem 1. Therefore P is a finitely generated free D-module and $End_D(P,P) \cong M_n(D^\circ) \cong (M_n(D))^\circ$ for some positive integer n. Thus A is isomorphic to $M_n(D)$. But A and D are finitely generated free R-modules by Theorem of [6]. Therefore n is uniquely determined, since such D is unique and since commutative rings have the invariant dimension property.

References

1. M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Tran. Amer. Math. Soc. **97** (1960), 367–409.
2. H. Bass, *K-theory and stable algebra*, Publ. Math. I.H.E.S. **22** (1964), 5-60.
3. F. R. DeMeyer, *Projective modules over central separable algebras*, Can. J. Math. **21** (1969), 39-43.
4. ______ and E. C. Ingraham, *"Separable algebras over commutative rings"*, Lecture Notes in Mathematics 181, Springer-Verlag, Berlin and New York, 1971.
5. D. R. Estes and R. M. Guralnick, *Module equivalences : Local to global when primitive polynomials represent units*, J. Algebra **77** (1982), 138-157.

6. B. R. McDonald and W. C. Waterhouse, *Projective modules over rings with many units*, Proc. Amer. Math. Soc. **83** (1981), 455-458.

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