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BERGMAN KERNEL FUNCTIONS

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1. Introduction

In this article, we study the Bergman kernel functions. In setion 2 we state $\mathcal{D}$ and $\mathcal{D}'$ be two bounded domains in $\mathcal{C}^n$ and let g, g' be the Bergman metrics on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then each holomorphic diffeomorphism of $\mathcal{D}$ onto $\mathcal{D}'$ is an isometry. In section 3 we study Bergman kernels for classical domanins computed by Hua([2],pp77-88). In section 4 we compute the Bergman kernel function for the polydisc and the balls. Consequently we have two Theorems The one is about the Bergman Kernel Function $\mathcal{K}(z,\zeta)$ for $\mathcal{B}$ and the Kähler form ω of the Bergman metric on $\mathcal{B}$ which $\mathcal{B} = \{(z_1, \cdots, z_n) \in \mathcal{C}^n | \sum_{j=1}^n |z_j|^2 < \mathcal{R}^2\}$ be the ball with radius $\mathcal{R} > 0$. The other is about the Bergman Kernel Function $\mathcal{K}(z,\zeta)$ for $\mathcal{D}$ and the Kähler form ω of the Bergman metric on $\mathcal{D}$ which $\mathcal{D} = \{(z_1, \cdots, z_n) \in \mathcal{C}^n ||z_j| < \mathcal{R}_j\}$ be the polydisc in $\mathcal{C}^n$.

2. The Bergman Metric

Let $\mathcal{D}$ be a bounded domain in $\mathcal{C}^n$. Let $\mathcal{L}^2(\mathcal{D})$ be the Hilbert space of all square integrable complex functions on $\mathcal{D}$. Let $\mathcal{H}(\mathcal{D})$ be the set of functions in $\mathcal{L}^2(\mathcal{D})$ which are holomorphic in $\mathcal{D}$. The inner product $(\ ,\ )$ on $\mathcal{L}^2(\mathcal{D})$ is given by

$$(2.1) \qquad (f.g) = \int_{\mathcal{D}} f(z)\overline{g(z)}d\mu, \qquad f,g \in \mathcal{L}^2(\mathcal{D}),$$

where $d\mu$ denote the Lebesque measure on $\mathcal{R}^{2n}$. We set $\|f\| = (f,g)^{\frac{1}{2}}$ for $f \in \mathcal{L}^2(\mathcal{D})$.

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THEOREM 2.1. *(Helgason [1],p365): $\mathcal{H}(\mathcal{D})$ is a closed subspace of $\mathcal{L}^2(\mathcal{D})$ and hence $\mathcal{H}(\mathcal{D})$ is a separable Hilbert space with the inner product (1.1).*

THEOREM 2.2. *(Helgason [1],pp365-367): Let $\varphi_0, \varphi_1, \cdots$ be a complete orthonormal system of the Hilbert space $\mathcal{H}(\mathcal{D})$. Then*

(i) *The series $\mathcal{K}(z,\zeta) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(\zeta)}$ converges uniformly on each compact subset of $\mathcal{D} \times \mathcal{D}$*

(ii) *$\mathcal{K}(z,\zeta)$ is independent of the choice of orthonormal basis $\{\varphi_k\}$.*

(iii) *For each $f \in \mathcal{H}(\mathcal{D})$,*

$$f(z) = \int_{\mathcal{D}} \mathcal{K}(z,\zeta)f(\zeta)d\mu(\zeta)$$

According to (iii) of Theorem 2.2,$\mathcal{K}(z,\zeta)$ is called the Bergman kernel function for a bounded domain $\mathcal{D}$

LEMMA 2.3. *For each $z \in \mathcal{D}$, $\mathcal{K}(z,z) > 0$. Therefore $log\mathcal{K}(z,z)$ is well-defined.*

proof. Let $\varphi_0, \varphi_1, \cdots$ be a complete orthonormal system of $\mathcal{H}(\mathcal{D})$. Then we clearly have

$$\mathcal{K}(z,z) = \sum_{k=0}^{\infty} \varphi_k(z)\overline{\varphi_k(z)} \geq 0.$$

Assume $\mathcal{K}(z_0,z_0) = 0$ for some $z_0 \in \mathcal{D}$. Since

$$\mathcal{K}(z_0,z_0) = \sum_k |\varphi_k(z_0)|^2 = 0, \text{ each } \varphi_k(z_0) = 0.$$

Thus for each $z \in \mathcal{D}$

$$\mathcal{K}(z_0,z) = \sum_k \varphi_k(z_0)\overline{\varphi_k(z)} = 0.$$

Hence for each $f \in \mathcal{H}(\mathcal{D})$, by (iii) of Theorem 2.2,

$$f(z_0) = \int_{\mathcal{D}} \mathcal{K}(z_0, z) f(z) d\mu(z) = 0.$$

This is contradiction. Therefore $\mathcal{K}(z,z) > 0$ for all $z \in \mathcal{D}$ and so $log\mathcal{K}(z,z)$ is well-defined. □

Let $z \in \mathcal{D}$ and$(z_1, \cdots, z_n)$ denote the components of z. Consider the complex tensor field h on $\mathcal{D}$ given by

$$(2.2) h = \sum_{1 \leq i,j \geq n} \frac{\partial^2 log\mathcal{K}(z,z)}{\partial z_i \partial \overline{z}_j} dz_i \otimes d\overline{z_j}.$$

If $\mathcal{X}$ is a real vector field on $\mathcal{D}$, that is, $\mathcal{X} = \sum_{i=1}^{n} (\zeta - i \frac{\partial}{\partial z_i} + \overline{\zeta_i} \frac{\partial}{\partial z_i})$, then we have

$$h(\mathcal{X}, \mathcal{X}) = \frac{1}{2\mathcal{K}^2} \sum_{k,l=0}^{\infty} |\sum_{i=0}^{\mathcal{N}} (\varphi_k \frac{\partial \varphi_l}{\partial z_i} - \varphi_l \frac{\partial \varphi_k}{\partial z_i}) \zeta_i|^2 \geq 0,$$

where $\varphi_0, \varphi_1, \cdots$ is a complete orthonormal system of $\mathcal{H}(\mathcal{D})$.It is known that h is a positive definite hermitian form on $\mathcal{T}_z(\mathcal{D}) \times \mathcal{T}_z(\mathcal{D})$. Let g be thereal part of h. Then

$$2g(\mathcal{X}, \mathcal{Y}) = h(\mathcal{X}, \mathcal{Y}) + \overline{h(\mathcal{X}, \mathcal{Y})} = h(\mathcal{X}, \mathcal{Y}) + h(\mathcal{Y}, \mathcal{X}), \quad \mathcal{X}, \mathcal{Y} \in \mathcal{T}(\mathcal{D}).$$

In addition, since

$$h(\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}) = h(\mathcal{X}, \mathcal{Y}), g(\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) \text{ for } \mathcal{X}, \mathcal{Y} \in \mathcal{T}(\mathcal{D}).$$

Hence $\mathcal{J}$ is an almost complex structure on $\mathcal{D}$. Hence $g = Reh$ is a Hermitian metric on $\mathcal{D}$.

If we set

$$\omega = Im\, h = \frac{1}{\sqrt{-1}} \partial \overline{\partial} log\mathcal{K}(z,z)$$
$$= \frac{1}{\sqrt{-1}} \sum_{i,j} \frac{\partial^2 log\mathcal{K}(z,z)}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z_j},$$

then

$$d\omega = \frac{1}{\sqrt{-1}} (\partial + \overline{\partial}) \partial \overline{\partial} log\mathcal{K}(z,z)$$
$$= \frac{1}{\sqrt{-1}} (\partial^2 \overline{\partial} - \partial \overline{\partial^2}) log\mathcal{K}(z,z) = 0$$

Since the second fundamental form ω is closed, $g = Reh$ is a Kähler metric on $\mathcal{D}$. This metric g is called the *Bergman metric* on $\mathcal{D}$.

Let φ be a holomorphic diffeomorphism of a bounded domain $\mathcal{D} \subset \mathcal{C}^n$ onto a bounded domain $\mathcal{D}' \subset \mathcal{C}^n$. Locally we have

$$\varphi(z_1, z_2, \cdots, z_n) = (\omega_1(z_1, \cdots, z_n), \cdots, \omega_n(z_1, \cdots, z_n)).$$

Then the Jacobian of φ

$$\mathcal{J}_\varphi = \frac{\partial(\omega_1 \cdots, \omega_n)}{\partial(z_1, \cdots, z_n)}$$

is a holomorphic function on $\mathcal{D}$. For the real coordinates given by $z_j = x_j + iy_j, \omega j = u_j + iv_j (1 \le j \le n)$, we have

$$|\mathcal{J}_\varphi|^2 = \frac{\partial(u_1, v_1, \cdots, u_n, v_n)}{\partial(x_1, y_1, \cdots, x_n, y_n)}.$$

Let μ and μ' denote the Lebesque measure on $\mathcal{D}$ and $\mathcal{D}'$ respectively.

Then

$$\mu'(\varphi(\mathcal{D})) = \int_{\mathcal{D}} |\mathcal{J}_\varphi|^2 d\mu.$$

Consequently, the mapping $\tilde{\varphi} : f \mapsto (f \circ \varphi)\mathcal{J}_\varphi$ is an isometry of $\mathcal{H}(\mathcal{D}')$, onto $\mathcal{H}(\mathcal{D})$. Let $\{\varphi'_k\}_{k=0}^\infty$ be a complete orthonormal system of $\mathcal{H}(\mathcal{D}')$, and then $\{\varphi_k = \tilde{\varphi}(\varphi'_k)\}_{k=0}^\infty$ is also a complete orthonormal system of

$$\begin{aligned}
\mathcal{K}_{\mathcal{D}}(z, \zeta) &= \sum_{k=0}^{\infty} \tilde{\varphi}(\varphi'_k)(z)\overline{\tilde{\varphi}(\varphi'_k)(z)} \\
&= \sum_{k=0}^{\infty} \varphi'_k(\varphi(z))\overline{\varphi'_k(\varphi(\zeta))}\mathcal{J}_\varphi(z)\overline{\mathcal{J}_\varphi(\zeta)} \\
&= \mathcal{K}_{\mathcal{D}'}(\varphi(z), \varphi(\varsigma))\mathcal{J}_\varphi(z)\overline{\mathcal{J}_\varphi(\zeta)}, \quad z, \zeta \in \mathcal{D}.
\end{aligned}$$

In particular , if $z \in \mathcal{D}$

$$\mathcal{K}_{\mathcal{D}}(z, z) = \mathcal{K}_{\mathcal{D}'}(\varphi(z), \varphi(z))|\mathcal{J}_\varphi(z)|^2. \tag{2.3}$$

Without proof, we state

THEOREM 2.4. *(Helgason [1],p370) : Let $\mathcal{D}$ and $\mathcal{D}'$ be two bounded domains in C^n and Let g,g' be the Bergman matrics on $\mathcal{D}$ and $\mathcal{D}'$ respectively. Then each holomorphic deffeomorphism of $\mathcal{D}$ and $\mathcal{D}'$ is* an isometry.

3. Bergman Kernels for Classical Domains

In 1935, E.carton proved that there exist only six types of irreducible homogeneous bounded symmetric domains.

$$TypeI:$$
$$\mathcal{D}_1 = \{Z \in \mathcal{M}_{m,n}(\mathcal{C}) | \mathcal{I}_m - Z^t\overline{Z} > 0\}.$$
$$TypeII:$$
$$\mathcal{D}_2 = \{Z \in \mathcal{M}_n(\mathcal{C}) |^t Z = Z, \mathcal{I}_n - Z\overline{Z} > 0\}.$$
$$TypeIII:$$
$$\mathcal{D}_3 = \{Z \in \mathcal{M}_n(\mathcal{C}) | Z +^t Z = 0, \quad \mathcal{I}_n + Z\overline{Z} > 0\}.$$

Type IV:

$$\mathcal{D}_4 = \{Z = (z_1, \cdots, z_n) \in \mathcal{C}_n | \quad |Z^tZ|^2 + 1 - 2Z^tZ > 0, \\ |Z^tZ| < 1\},\ n > 2$$

This above four bounded symmetric domains are called *classical domains.* Beside the above four types we have exceptional ones ; there dimensions are 16 and 27 . Hua([2],pp77-88) computed the Bergman kernel for the above four classical domains. We write down ones.

THEOREM 3.1. *(Hua[2]): The Bergman kernel $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4$, for $\mathcal{D}_1$,$\mathcal{D}_2$,$\mathcal{D}_3$,$\mathcal{D}_4$ are given respectively as follows·*

(I)

$$\mathcal{K}_1(Z, Z) = \frac{\{det(\mathcal{I}_m - Z^t\overline{Z})\}^{-(m+n)}}{vol(\mathcal{D}_1)},$$
$$\text{where } vol(\mathcal{D}_1) = \frac{1!2!\cdots(m-1)!1!2!\cdots(n-1)!}{1!2!\cdots(m+n-1)!}\pi^m n.$$

(II)

$$\mathcal{K}_2(Z,Z) = \frac{\{det(\mathcal{I}_n - Z\overline{Z})\}^{(n+1)}}{vol(\mathcal{D}_2)},$$

$$where\ vol(\mathcal{D}_2) = \pi^{\frac{n(n-1)}{2}} \cdot \frac{2!4!\cdots(2n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

(III)

$$\mathcal{K}_3(Z,Z) = \frac{\{det(\mathcal{I}_n + Z\overline{Z})\}^{-n+1}}{vol(\mathcal{D}_3)},$$

$$where\ vol(\mathcal{D}_3) = \pi^{\frac{n(n-1)}{2}} \cdot \frac{2!4!\cdots(2n-4)!}{(n-1)!n!\cdots(2n-3)!}.$$

(IV)

$$\mathcal{K}_4(Z,Z) = \frac{(1+|Z^tZ|^2 - 2\overline{Z}^tZ)^{-n}}{vol(\mathcal{D}_4)},$$

$$where\ vol(\mathcal{D}_4) = \frac{\pi^n}{2^{n-1}n!}$$

REMARK. *The problem of explicit description of the two exceptional cases is still open.*

4. Bergman Kernel Functions for the Balls

In this section, we compute the Bergman kernel functions for the polydisc and the ball.

LEMMA 4.1. *Let $a \in \mathcal{C}$ and $\mathcal{R} > 0$. Then we have*

$$\int_{|z-a|<\mathcal{R}} (z-a)^k\overline{(z-a)^l}d\mu = \begin{cases} 0, & if\ k \neq l \\ \frac{\pi\mathcal{R}^{2k+2}}{k+1}, & if\ k = l. \end{cases}$$

Here k,l denote the intezers and $d\mu$ denote the Lebesque measure on $\mathcal{R}^2$.

Proof.. We set $z - a = re^{i\theta}, 0 \leq r < \mathcal{R}, 0 \leq \theta < 2\pi$. Then we have

$$\int_{|z-a|<\mathcal{R}} (z-a)^k\overline{(z-a)^l}d\mu = \int_0^{\mathcal{R}}\int_0^{2\pi} r^{k+l+1}e^{i(k-l)}\,d\theta dr.$$

Thus we obtain the desired result. □

Let $\mathcal{B} = \{z = (z_j) \in \mathcal{C}^n \mid \sum_{j=1}^n, |z_j|^2 < \mathcal{R}^2\}$ be the ball radius $\mathcal{R}$. Now we compute the Bergman kernel function for $\mathcal{B}$.

LEMMA 4.2. *The monomials* $\varphi_{(\nu)}(z) = z_1^{\nu_1} \cdots z_n^{\nu_n} (\nu_j = 0, 1, 2, \cdots)$ *form a complete orthogonal system in* $\mathcal{H}(\mathcal{B})$ *but not normalized. Here we set* $(\nu) = (\nu_1, \cdots, \nu_n)$.

Proof. We first note that $\varphi_{(\nu)}(z) \neq \varphi_{(\mu)}(z)$ if and only if $\nu_k \neq \mu_k$ for some k. Say, $\mu_n \neq \nu_n$.

$$(\varphi_{(\nu)}, \varphi_{(\mu)}) = \int_{\mathcal{B}_{n-1}} z_1^{\nu_1} \cdots z_{n-1}^{\nu_{n-1}} \overline{z}_1^{\mu_1} \cdots \overline{z}_{n-1}^{\mu_{n-1}} d\mu_{n-1} \int_{|z_n|<\mathcal{R}} z_n^{\nu_n} \overline{z}_n^{\mu_n} d\mu_1,$$

where

$$\mathcal{B}_{n-1} = \{(z_1, \cdots, z_{n-1}) \in \mathcal{C}^n \mid \sum_{j=1}^{n-1} |z_j|^2 < \mathcal{R}^2 - |z_n|^2\}$$

and $d\mu_1, d\mu_{n-1}$ denotes the Lebesque measures on $\mathcal{R}^2$ and $\mathcal{R}^{2(n-1)}$ respectively. Since, by Lemma 4.1,

$$\int_{|z_n|<\mathcal{R}} z_n^{\nu_n} \overline{z}_k^{\mu_n} d\mu_1 = 0,$$

we have $(\varphi_{(\nu)}, \varphi_{(\mu)}) = 0$ for $\nu \neq \mu$. Therefore $\{\varphi_{(\nu)}(z)\}$ forms a complete orthogonal system in $\mathcal{H}(\mathcal{B})$. □

LEMMA 4.3.

$$\|\varphi(\nu)\|^2 = \pi^n \mathcal{R}^{2(n+1)} \frac{\nu_1! \cdots \nu_n!}{(\nu+n)!}, \quad \text{where } \nu = \sum_{j=1}^{n} \nu_j. \tag{4.1}$$

Proof. We prove this by induction on n. If n=1. (4.1) holds by Lemma 4.1.

$$\|\varphi(\nu)(z)\|^2 = \int_{|z_n|<\mathcal{R}} \left(\int_{\mathcal{B}_n} |z_1|^{2\nu_1} \cdots |z_{n-1}|^{2\nu_{n-1}} d\nu_{n-1}\right) |z_n|^{2\nu_n} d\nu_1$$

$$= \int_{|z_n|<\mathcal{R}} \frac{\pi^{n-1} (\mathcal{R}^2 - |z_n|^2)^{(n-1)+\sum_{j=1}^{n-1} \nu_j} \nu_1! \cdots \nu_n!}{(\nu_1 + \cdots + \nu_{n-1} + n - 1)!} |z_n|^{2\nu_n} d\mu_1,$$

where $\mathcal{B}_n = \{(z_1, \cdots, z_{n-1}) \in \mathcal{C}^{n-1} \mid \sum_{j=1}^{n-1} |z_j|^2 < \mathcal{R}^2 - |z_n|^2\}$.

We set $z_n = re^{i}\theta, 0 \le r < \mathcal{R}, 0 \le \theta < 2\pi$. Then

$$\begin{aligned}
&\|\varphi(\nu)(z)\|^2 \\
&= \frac{\pi^{n-1}\nu_1!\ldots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\int_0^{2\pi}\int_0^{\mathcal{R}}(\mathcal{R}^2 - r^2)^{(n-1)+\sum_{j=1}^{n-1}\nu_j} r^{2\nu_n} r dr d\theta \\
&= \frac{2\pi^{n-1}\nu_1!\cdots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\int_0^{\mathcal{R}}(\mathcal{R}^2 - r^2)^{(n-1)+\sum_{j=1}^{n-1}\nu_j} r^2\nu_n r dr.
\end{aligned}$$

We set $r^2 = \mathcal{R}^2 t (0 \le t < 1)$. Then we have

$$\begin{aligned}
\|\varphi_{(\nu)}\|^2 &= \frac{2\pi^n\nu_1!\cdots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\int_0^1 (1-t)^{(n-1)+\sum_{j=1}^{n-1}\nu_j}\mathcal{R}^{2(\nu+n)}\frac{t^{\nu_n}}{2}dt \\
&= \pi^n\mathcal{R}^{2(\nu+n)}\frac{\nu_1!\cdots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\int_0^1 (1-t)^{(n-1)+\sum_{j=1}^{n-1}\nu_j} t^{\nu_n} dt \\
&= \pi^n\mathcal{R}^{2(\nu+n)}\frac{\nu_1!\cdots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\mathcal{B}(\nu_n + 1, \sum_{j=1}^{n-1}\nu_j + n) \\
&= \pi^n\mathcal{R}^{2(\nu+n)}\frac{\nu_1!\cdots\nu_{n-1}!}{(\sum_{j=1}^{n-1}\nu_j + n - 1)!}\cdot\frac{\Gamma(\nu_n+1)\Gamma(\sum_{j=1}^{n-1}\nu_j + n)}{\Gamma(\nu+n)} \\
&= \pi^n\mathcal{R}^{2(\nu+n)}\frac{\nu_1!\cdots\nu_n!}{(\nu+n)!}.
\end{aligned}$$

Here $\mathcal{B}(u,v)$ is the beta function and we used the identy

$$\mathcal{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)},$$

where $\Gamma(u)$ denotes the Gamma function (see appendix). □

LEMMA 4.4. *Let* $f(z) = \left(1 - \frac{\sum_{j=1}^n z_j}{\mathcal{R}^2}\right)^{-(n+1)}$. *Then the Laurent Series of* $f(z)$ *is given by*

$$f(z) = \sum_\alpha c_\alpha z^\alpha = \sum_\alpha c_{(\alpha_1,\cdots,\alpha_n)} z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

where

$$c_\alpha = \frac{\mathcal{D}^\alpha f)(0)}{\alpha!} = \frac{(n+\alpha)}{(\mathcal{R}^{2\alpha}\cdot n!)\alpha_1!\cdots\alpha_n!}, \quad \alpha = \sum_{j=1}^{n}\alpha_j.$$

Proof. Obvious.

LEMMA 4.5. *The Bergman Kernel Function $\mathcal{K}(z,\zeta)$ is given by*

$$\mathcal{K}(z,\zeta) = \frac{n!\mathcal{R}^2}{\pi^n}(\mathcal{R}^2 - \sum_{j=1}^{n} z_j\overline{\zeta_j})^{-(n+1)}.$$

Proof.

$$\mathcal{K}(z,\zeta) = \sum_\nu \|\varphi_{(\nu)}\|^2\varphi_{(\nu)}(z)\overline{\varphi_{(\nu)}(\zeta)}$$

$$= \sum_\nu \frac{(\nu+n)!}{\pi^n\mathcal{R}^{2(n+\nu)}\nu_1!\cdots\nu_n!}(z_1\overline{\zeta_1})^{\nu_1}\cdots(z_n\overline{\zeta_n})^{\nu_n}, \quad \text{by Lemma 4.3}$$

$$= \frac{n!}{\pi^n\mathcal{R}^{2n}}\sum_\nu \frac{(\nu+n)!}{(\mathcal{R}^{2\nu}n!)\nu_1!\cdots\nu_n!}(z_1\overline{\zeta_1})^{\nu_1}\cdots(z_n\overline{\zeta_n})^{\nu_n}$$

$$= \frac{n!}{\pi^n\mathcal{R}^{2n}}\left(1 - \frac{\sum_{j=1}^n z_j\overline{\zeta_j}}{\mathcal{R}^2}\right)^{-(n+1)} \quad \text{by Lemma 4.4}$$

$$= \frac{n!\mathcal{R}^2}{\pi^n}(\mathcal{R}^2 - \sum_{j=1}^{n} z_j\overline{\zeta_j})^{-(n+1)}. \qquad \square$$

LEMMA 4.6.

$$\frac{\partial^2 \log\mathcal{K}(z,z)}{\partial z_j\partial\bar{z}_j} = (n+1)\frac{\bar{z}_i z_j + \delta_{ij}(\mathcal{R}^2 - \sum_{k=1}^n |z_k|^2)}{(\mathcal{R}^2 - \sum_{k=1}^n |z_k|^2)^2}.$$

Proof. Since

$$logK(z,z) = \log \frac{n!\mathcal{R}^2}{\pi^n} - (n+1)\log(\mathcal{R}^2 - \sum_{j=1}^{n} z_j\bar{z}_j),$$

$$\frac{\partial log\mathcal{K}(z,z)}{\partial z_j \partial \bar{z}_j} = \frac{(n+1)z_j}{\mathcal{R}^2 - \sum_{j=1}^{n}|z_j|^2}.$$

Hence

$$\frac{\partial^2 \log\mathcal{K}(z,z)}{\partial z_i \partial \bar{z}_j} = (n+1)\frac{\bar{z}_j z_j}{(\mathcal{R}^2 - \sum_{j=1}^{n}|z_j|^2)^2} \quad \text{if } i \neq j,$$

$$\frac{\partial^2 log\mathcal{K}(z,z)}{\partial z_j \partial \bar{z}_j} = (n+1)\frac{\bar{z}_j z_j + (\mathcal{R}^2 - \sum_{j=1}^{n}|z_j|^2)}{(\mathcal{R}^2 - \sum_{j=1}^{n}|z_j|^2)^2}.$$

Therefore we obtain the desired result. □

Let $\mathcal{D} = \{(z_1, \cdots, z_n) \in \mathcal{C}^n \,|\, |z_j| < \mathcal{R}_j\}(\mathcal{R}_j > 0)$ be a polydisc in $\mathcal{C}^n$. Let $\mathcal{D}_j = \{z \in \mathcal{C} \,|\, |z| < \mathcal{R}_j\}(1 \leq j \leq n)$. Then $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$. By Lemma 4.5 , the Bergman Kernel Function $\mathcal{K}_j(z,\zeta)$ for $\mathcal{D}_j$ is given by

$$\mathcal{K}_j(z,\zeta) = \frac{\mathcal{R}_j^2}{\pi}(\mathcal{R}_j^2 - z\bar{\zeta})^{-2}, \quad (z,\zeta) \in \mathcal{D}_j \times \mathcal{D}_j.$$

Thus the Bergman Kernel Function $\mathcal{K}(z,\zeta)$ for $\mathcal{D}$ is

$$\begin{aligned} \mathcal{K}(z,\zeta) &= \prod_{j=1}^{n} \mathcal{K}_j(z_j,\zeta_j) \\ &= \frac{1}{\pi^n}\prod_{j=1}^{n} \mathcal{R}_j^2(\mathcal{R}_j^2 - z_j\zeta_j)^{-2}, \end{aligned}$$

where $z = (z_1, \cdots, z_n) \in \mathcal{D}$ and $\zeta = ((\zeta_1, \cdots, \zeta_n) \in \mathcal{D}$. Since

$$\log\mathcal{K}(z,z) = -\log\pi^n + \sum_{j=1}^{n}\log\mathcal{R}_j^2 - 2\sum_{j=1}^{n}\log(\mathcal{R}_j^2 - z_j\bar{z}_j),$$

we have

$$\frac{\partial\log\mathcal{K}(z,z)}{\partial\bar{z}_j} = \frac{2z_j}{\mathcal{R}^2 - z_j\bar{z}_j}$$

Thus

$$\frac{\partial^2 log\mathcal{K}(z,z)}{\partial z_i \partial \bar{z}_j} = 0 \quad \text{if } i \neq j$$

and

$$\frac{\partial^2 \log \mathcal{K}(z,z)}{\partial z_j \partial \bar{z}_j} = \frac{2\mathcal{R}_j^2}{\mathcal{R}_j^2 - |z_j|^2}.$$

In summary, we have

THEOREM 4.7. *Let $\mathcal{B} = \{(z_1, \cdots, z_n) \in \mathcal{C}^n | \sum_{j=1}^n |z_j|^2 < \mathcal{R}^2\}$ be the ball with radius $\mathcal{R} > 0$. Then the Bergman Kernel Function $\mathcal{K}(z,\zeta)$ for $\mathcal{B}$ is given by*

$$\mathcal{K}(z,\zeta) = \frac{n!\mathcal{R}^n}{\pi^n}(\mathcal{R}^2 - \sum_{j=1}^n z_j\bar{\zeta}_j)^{-(n+1)}, \ (z,\zeta) \in \mathcal{B} \times \mathcal{B}.$$

The Kähler form ω of the Bergman metric on $\mathcal{B}$ is

$$\omega = \frac{n+1}{\sqrt{-1}} \sum_{i,j} \frac{\bar{z}_i z_j + \delta_{ij}(\mathcal{R}^2 - \sum_{k=1}^n |z_k|^2)}{(\mathcal{R}^2 - \sum_{k=1}^n |z_k|^2)^2} dz_j \wedge d\bar{z}_j.$$

THEOREM 4.8. *Let $\mathcal{D} = \{(z_1, \cdots, z_n) \in \mathcal{C}^n | |z_j| < \mathcal{R}_j\}$ be the polydisc in $\mathcal{C}^n$. Then the Bergman Kernel Function $\mathcal{K}(z,\zeta)$ for $\mathcal{D}$ is given by*

$$\mathcal{K}(z,\zeta) = \frac{1}{\pi^n} \prod_{j=1}^n \mathcal{R}_j^2(\mathcal{R}_j^2 - z_j\bar{\zeta}_j)^{-2},$$

where $z = (z_1, \cdots, z_n) \in \mathcal{D}$ and $\zeta = (\zeta_1, \cdots, \zeta_n) \in \mathcal{D}$. The Kähler form ω of the Bergman metric on $\mathcal{D}$ is

$$\omega = \frac{2}{\sqrt{-1}} \sum_{j=1}^n \frac{z_j}{\mathcal{R}_j - z_j\bar{z}_j} dz_j \wedge d\bar{z}_j.$$

Appendix. The gamma function $\Gamma(u)$ is defined by

$$\Gamma(u) = \int_0^\infty x^{u-1}\exp(-x)dx$$
$$= 2^{1-u}\int_0^\infty s^{2u-1} exp(-\frac{s^2}{2})ds, \quad u > 0.$$

the beta function $\mathcal{B}(u,v)$ is defined by

$$\mathcal{B}(u,v) = 2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2u-1}(\sin\theta)^{2u-1}d\theta,$$
$$= \int_0^1 z^{u-1}(1-z)^{u-1}dz, \quad u>0, v>0$$

$$\Gamma(u)\Gamma(v) = 2^{2-u-v}\int_0^\infty s^{2u-1}exp(-\frac{s^2}{2})ds\int_0^\infty t^{2v-1}exp(-\frac{t^2}{2})dt$$
$$= 2^{2-u-v}\int_0^\infty r^{2(u+v)-1}exp(-\frac{r^2}{2})dr\int_0^{\frac{\pi}{2}}(cos\theta)^{2u-1}(sin\theta)^{2v-1}d\theta$$
$$= \Gamma(u+v)\mathcal{B}(u,v).$$

Thus we obtain

$$\mathcal{B}(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad u>0, v>0.$$

Now we compute the volume α_n of the unit n-ball $\mathcal{B}$ in $\mathcal{R}^n$,

$$\mathcal{B} = \{(x_1,\cdots,x_n)\in\mathcal{R}^n|\sum_{j=1}^n x_j^2 < 1\}$$

$$\alpha_n = \int_{\mathcal{B}} d\mu_n$$
$$= \int_{x_n^2\le 1}\left(\int_{x_1^2+\cdots+x_{n-1}^2\le 1-x_n^2} d\mu_{n-1}\right)d\mu$$
$$= \int_{x_n^2\le 1}(1-x_n^2)^{\frac{n-1}{2}}\alpha_{n-1}d\mu_1$$
$$= 2\alpha_{n-1}\int_0^1(1-u^2)^{\frac{u-1}{2}}du.$$

We set $u=\sqrt{z}$, and then $du = \frac{dz}{2\sqrt{z}}$.

Hence

$$2\int_0^1 (1-u^2)^{\frac{n-1}{2}} du = \int_0^1 z^{-\frac{1}{2}}(1-z)^{\frac{n-1}{2}} dz$$
$$= \mathcal{B}(\frac{1}{2}, \frac{n+1}{2}).$$

Thus

$$\alpha_n = \alpha_{n-1}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2}+1)}.$$

Moreover, $\alpha_1 = 2$. By induction on n,

$$\alpha_n = \frac{\pi^{\frac{n}{2}}}{(\frac{n}{2})\Gamma(\frac{n}{2})}, \;\; n = 1, 2, \cdots.$$

If $n = 2l$ is even, $\alpha_{2l} = \frac{\pi^l}{l!}$.

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