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PROPERTY *(P)* ON ℓ_p

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X and Y are real Banach spaces with closed unit balls B_X and B_Y respectively. For $n = 0, 1, \dots,$ the space $\mathcal{P}({}^nX, Y)$ of contiuous n–*homogeneous polynomials* $P : X \to Y$ consists of all functions P of the form $P(x) = A(x, \dots, x)$, where $A : X \times \cdots \times X \to Y$ is a continuous n–linear mapping. $\|P\| \equiv \sup\{\|P(x)\| \ : \ x \in B_X\}$. The space $P(X, Y)$ is the algebraic direct sum of the space $\mathcal{P}({}^nX, Y), n = 0, 1, 2, \dots.$ $\mathcal{P}(X)$ and $\mathcal{P}({}^nX)$ denote $\mathcal{P}(X, \mathbb{R})$ and $\mathcal{P}({}^nX, \mathbb{R})$, respectively.

We say that a Banach space X has *property (P)* if for any bounded sequences (u_n) and (v_n) in X such that $|P(u_n) - P(v_n)| \to 0$ as $n \to \infty$ for all $P \in \mathcal{P}({}^nX)$, $n \geq 1$, then $|Q(u_n - v_n)| \to 0$ as $n \to \infty$ for all $Q \in \mathcal{P}({}^nX), n \geq 1$. This property was studied in [ACL], which is closely related with Dundord–Pettis property and Schur property. Aron, Choi and Llavona [ACL] showed that every super-reflexive Banach space has *property (P)*. However in their proof they used the fact that every super–reflexive Banach space is in W_p–class, which was studied by Castillo and Sánchez[CS]. Hence we cannot have the exact form of polynomial which works in their proof. In this note we will prove that every ℓ_p $(1 \leq p < \infty)$ has *property (P)*, without using W_p–class. For general background on polynomials we refer to [D] and [M].

Lemma 1. *For any p, $1 \leq p < \infty$, if (u_j) and (v_j) are two sequences in ℓ_p which go to 0 weakly, and if for all polynomials $P \in \mathcal{P}(\ell_p)$, $P(u_j) - P(v_j) \to 0$, then $\|u_j - v_j\|_p \to 0$*

Proof. It is enough to prove the case $1 < p < \infty$. Suppose that $\|u_j - v_j\|_p \not\to 0$. Since (u_j) and (v_j) converge to 0 weakly, by passing to a subsequence, there is an increasing sequence (n_j) of positive integers such that for $E_j = \{n_j \leq k \leq n_{j+1} - 1\}$,

$$\|u_j \chi_{E_j} - u_j\|_p < \frac{1}{2^j} \quad \text{and} \quad \|v_j \chi_{E_j} - v_j\|_p < \frac{1}{2^j}.$$

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We may assume (u_j) and (v_j) to be $(u_j\chi_{E_j})$ and $(v_j\chi_{E_j})$ respectively. Let

$$E_j^+ = \{k \in E_j : u_j^k v_j^k \geq 0\}, \quad \text{and} \quad E_j^- = \{k \in E_j : u_j^k v_j^k < 0\},$$

where $u_j = (u_j^1, u_j^2, \dots, u_j^n, \dots)$ and $v_j = (v_j^1, v_j^2, \dots, v_j^n, \dots)$. Since $\|u_j - v_j\|_p \not\to 0$. We have that $\|(u_j - v_j)\chi_{E_j^+}\|_p \not\to 0$ or $\|(u_j - v_j)\chi_{E_j^-}\|_p \not\to 0$. When $\|(u_j - v_j)\chi_{E_j^+}\|_p \not\to 0$, we partition each set E_j^+ into four pairwise disjoint subsets. $P_{u_j}, P_{v_j}, N_{u_j}, N_{v_j}$ where

$$\begin{aligned} P_{u_j} &= \{k \in E_j^+ : u_j^k \geq v_j^k > 0\}, \\ P_{v_j} &= \{k \in E_j^+ : v_j^k > u_j^k > 0\}, \\ N_{u_j} &= \{k \in E_j^+ : u_j^k \leq v_j^k \leq 0\} \text{ and} \\ N_{v_j} &= \{k \in E_j^+ : v_j^k < u_j^k \leq 0\}. \end{aligned}$$

Since $\|(u_j - v_j)\chi_{E_j^+}\|_p \not\to 0$, one of the following four sequences does not converge to 0; $(\|(u_j - v_j)\chi_{P_{u_j}}\|_p)$, $(\|(u_j - v_j)\chi_{P_{v_j}}\|_p)$, $(\|(u_j - v_j)\chi_{N_{u_j}}\|_p)$, $(\|(u_j - v_j)\chi_{N_{v_j}}\|_p)$. Suppose $(\|(u_j - v_j)\chi_{P_{u_j}}\|_p)$ does not converge to 0. We may assume that for each j, $\|(u_j - v_j)\chi_{P_{u_j}}\|_p \geq \delta$ for some $\delta > 0$ (consider a subsequence if necessary). Define

$$P(x) = \sum_{j=1}^{\infty} \left(\sum_{k \in P_{u_j}} x^k |u_j^k - v_j^k|^{p-1} \right)^N$$

where N is an integer greater than p and $x = (x^1, x^2, \dots, x^n, \dots) \in \ell_p$. If $\|x\|_p \leq 1$, then

$$\begin{aligned} |P(x)| &\leq \sum_{j=1}^{\infty} \left(\sum_{k \in P_{u_j}} |x^k| |u_j^k - v_j^k|^{p-1} \right)^N \\ &\leq \sum_{j=1}^{\infty} \left(\sum_{k \in P_{u_j}} |x^k|^p \right)^{\frac{N}{p}} \left(\sum_{k \in P_{u_j}} |u_j^k - v_j^k|^{(p-1)q} \right)^{\frac{N}{q}} \end{aligned}$$

(q is the exponential conjugate of p).

Since (u_j) and (v_j) are bounded sequences in ℓ_p, there exists a constant $C > 0$ such that

$$\left(\sum_{k\in P_{u_j}} |u_j^k - v_j^k|^{(p-1)q}\right)^{\frac{N}{q}} = \left(\sum_{k\in P_{u_j}} |u_j^k - v_j^k|^p\right)^{\frac{N}{q}} \le C$$

for every j. Thus

$$|P(x)| \le C\sum_{j=1}^{\infty} \left(\sum_{k\in P_{u_j}} |x^k|^p\right)^{\frac{N}{p}} \le C\|x\|_p^p \le C,$$

for $\|x\|_p \le 1$. The second inequality above comes from the fact $\frac{N}{p} \ge 1$ and $\|x\|_p \le 1$. Thus P is a continuous N-homogeneous polynomial on ℓ_p. However

$$\begin{aligned} &P(u_\ell) - P(v_\ell) \\ &= \left(\sum_{k\in P_{u_\ell}} u_\ell^k |u_\ell^k - v_\ell^k|^{p-1}\right)^N - \left(\sum_{k\in P_{u_\ell}} v_\ell^k |u_\ell^k - v_\ell^k|^{p-1}\right)^N \\ &\ge \left(\sum_{k\in P_{u_\ell}} |u_\ell^k - v_\ell^k|^p\right)^N = \|(u_\ell - v_\ell)\chi_{P_{u_\ell}}\|_p^{pN} \ge \delta^{pN}, \end{aligned}$$

which contradicts hypothesis. (The first inequality above from the fact that if $a \ge b > 0$, then $(a-b)^N \ge a^N - b^N$.) The other cases are proved similarly to the above.

On the other hand, when $\|(u_j - v_j)\chi_{E_j^-}\|_p \not\to 0$, we have that $\|u_j\chi_{E_j^-}\|_p \not\to 0$ or $\|v_j\chi_{E_j^-}\|_p \not\to 0$. We may assume without loss of generality that for each j, $\|u_j\chi_{E_j^-}\|_p \ge \delta$ for some $\delta > 0$ (consider a subsequence if necessary). Let $z_j = \frac{u_j\chi_{E_j^-}}{\|u_j\chi_{E_j^-}\|_p}$ and then (z_j) is a normalized basic sequence in ℓ_p. Let Z be the closed subspace of ℓ_p

spanned by (z_j). Define $\pi : \ell_p \to \ell_p$ by

$$\pi(x) = \sum_{j=1}^{\infty} \left(\sum_{k \in E_j^-} x^k (\text{sign } z_j^k) |z_j^k|^{p-1} \right) z_j$$
$$(x = (x^1, x^2, \dots) \in \ell_p).$$

Since (z_j) is a normalized sequence with pairwise disjoint supports,

$$\begin{aligned}\|\pi(x)\|_p &= \left(\sum_{j=1}^{\infty} \left| \sum_{k \in E_j^-} x^k (\text{sign } z_j^k) |z_j^k|^{p-1} \right|^p \right)^{\frac{1}{p}} \\ &\le \left[\sum_{j=1}^{\infty} \left(\sum_{k \in E_j^-} |x^k|^p \right) \left(\sum_{k \in E_j^-} |z_j^k|^{(p-1)q} \right)^{\frac{p}{q}} \right]^{\frac{1}{p}}\end{aligned}$$

Since $\sum_{k \in E_j^-} |z_j^k|^{(p-1)q} = \sum_{k \in E_j^-} |z_j^k|^p = 1$ for every j, we obtain $\|\pi(x)\|_p \le \|x\|_p$ and also $\pi(z_\ell) = z_\ell$ for every ℓ. Hence π is a norm 1 projection from ℓ_p onto Z.

Define $P : Z \to \mathbf{R}$ by $P(\sum a_j z_j) = \sum a_j^N$ where N is an odd integer greater than p. It is easy to see $P \in \mathcal{P}(^N Z)$ and hence $\tilde{P} = P \circ \pi \in \mathcal{P}(^N \ell_p)$. For each ℓ, we get

$$\begin{aligned}\tilde{P}(u_\ell) &= (P \circ \pi)(u_\ell) \\ &= \left(\sum_{k \in E_\ell^-} u_\ell^k (\text{sign } z_\ell^k) |z_\ell^k|^{p-1} \right)^N \\ &= \left(\|u_\ell \chi_{E_\ell^-}\|_p \sum_{k \in E_\ell^-} |z_\ell^k|^p \right)^N \\ &= \|u_\ell \chi_{E_\ell^-}\|_p^N \ge \delta^N\end{aligned}$$

and

$$\tilde{P}(v_\ell) = \left(\sum_{k \in E_\ell^-} v_\ell^k (\text{sign } z_\ell^k) |z_\ell^k|^{p-1} \right)^N < 0.$$

This implies $\tilde{P}(u_\ell) - \tilde{P}(v_\ell) \geq \delta^N$ for every ℓ, which contradicts hypothesis. □

THEOREM 2. *For any $1 \leq p \leq \infty$, ℓ_p has property (P).*

Proof. We only need to consider $p \in (1, \infty)$, since ℓ_1 and ℓ_∞ have the Dunford Pettis property. Let (u_n) and (v_n) be bounded sequences in X such that $|P(u_n) - P(v_n)| \to 0$ for all polynomials P. Using the reflexivity of ℓ_p, we may suppose without loss of generality that both (u_n) and (v_n) tend weakly to some $x \in \ell_p$. Moreover, our hypothesis implies that for all continuous polynomials P, $P(u_n - x) - P(v_n - x) \to 0$. To see this, let A be the continuous symmetric k-linear form associated with a continuous k-homogeneous polynomial P_k. Thus

$$\begin{aligned}
&P_k(u_n - x) - P_k(v_n - x) \\
=&A(u_n - x, ..., u_n - x) - A(v_n - x, ..., v_n - x) \\
=&[A(u_n, ..., u_n) - A(v_n, ..., v_n)] + k[A(x, u_n, ..., u_n) - A(x, v_n, ..., v_n)] \\
&+ \cdots + k[A(x, ..., x, u_n) - A(x, ..., x, v_n)] \\
=&[P_k(u_n) - P_k(v_n] + k[P_{k-1}(u_n) - P_{k-1}(v_n)] \\
&+ \cdots + k[P_1(u_n) - P_1(v_n)],
\end{aligned}$$

where P_i is the i-homogeneous polynomial defined by $P_i(y) \equiv A(x^{k-i}y^i)$, $(1 \leq i \leq k)$.

Therefore, the sequences $(u_n - x)$ and $(v_n - x)$ satisfy the conditions of the preceding lemma. Hence $\|(u_n - x) - (v_n - x)\| = \|u_n - v_n\| \to 0$. Thus, for every $Q \in \mathcal{P}(^nX), n \geq 1$, $Q(u_n - v_n) \to 0$. □

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