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ENDOMORPHISM RINGS OF ARTINIAN MODULES*

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Endomorphism rings of Artinian modules need not to be semiperfect by a result of Camps and Menal [CM], which answering in the negative to a question of Crawley and Jonsson [CJ].

In spite of this fact, related to a question [M, Question 16], we mainly observe, in this paper, endomorphism rings of Artinian modules over a certain class of PI–rings. Explicitly we show that endomorphism rings of such Artinian modules are semilocal, thereby we can provide a partial affirmative answer to Question 16 in [M]. Also as a byproduct, an interesting result is that stable range of such endomorphism rings are 1.

Recall that a ring R is called *semilocal* if $R/J(R)$ is a right Artinian ring, where $J(R)$ is the Jacobson radical of R. A semilocal ring R is called *semiprimary* if $J(R)$ is nilpotent

For an Artinian R–module M with the endomorphism ring S, let $N(S)$ be the ideal of endomorphisms of M whose kernels are essential in M. Then, if M is Artinian, we have $N(S) \subseteq J(S)$, where $J(S)$ is the Jacobson radical of the ring S. In fact, let s be in $N(S)$. Then $Ker(s)$ is essential in M. Thus from the fact that $Ker(s) \cap Ker(1-s) = 0$, we have $Ker(1-s) = 0$ and so $1-s$ is an isomorphism because M is an Artinian module. Therefore $1-s$ is invertible in S for any s in $N(S)$. Hence $N(S) \subseteq J(S)$.

An overring B of R is called an *extension* if $B = RC_B(R)$, where $C_B(R) = \{b \in B \mid br = rb \text{ for all } r \in R\}$, the *centralizer* of R in B. For an example, when R is a central subring of B, B is an extension of R. In the sense of Schelter [S], an overring B of R is *integral* over R if for every $b \in B$, we have $b^n + r_{n-1}b^{n-1} + \ldots + r_0 = 0$ or $b^n + b^{n-1}r_{n-1} + \ldots + r_0 = 0$ for some $r_i \in R$ and some positive integer n. By [PS], every ring which is finitely generated as a module over its central subring C is an integral PI–extension of C.

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Related to Hilbert Nullstellensatz, recall that a commutative ring is called a *Jacobson ring* if every prime ideal of R is an intersection of maximal ideals. Also a commutative domain K is called a *G-domain* if $M \cap K = 0$ for some maximal ideal M of the polynomial ring $K[x]$. A prime ideal P of a commutative ring K is said to be a *G-ideal* if the ring K/P is a G-domain. Note that in a Jacobson ring, every G-ideal is a maximal ideal.

Theorem 1. *Assume that R is a PI-ring which is either integral over its center or an affine algebra over a commutative Jacobson ring. Then the endomorphism ring $End_R(M)$ of an Artinian R-module M is semilocal.*

Proof. Case I. Assume that R is a PI-ring which is integral over its center. Let $Z(R)$ be the center of R and $S = End_R(M)$. Write $Soc(M) = H_1 \oplus H_2 \oplus \cdots \oplus H_t$, where the H_i's represent the homogeneous components of $Soc(M)$. Let $P_i = Ann_R(H_i)$, the annihilator of H_i in R; then each P_i is a primitive ideal and $A = \cap P_i = Ann_R(Soc(M))$.

Let $C_i = Z(R/P_i)$, the center of R/P_i, which is a field. Because M is Artinian, each H_i is finitely generated R-module, hence a finitely generated R/P_i-module. Since R/P_i is a primitive PI-ring, R/P_i is finite dimensional over its center C_i and so each H_i is finite dimensional over C_i. Therefore $Soc(M) = H_1 \oplus H_2 \oplus \cdots \oplus H_t$ is a finitely generated module over $C = C_1 \oplus C_2 \oplus \cdots \oplus C_t = Z(R/A)$, the center of R/A. Note that $R/A = \bigoplus_{i=1}^t (R/P_i)$ because each P_i is a maximal ideal of R. Now $End_R(Soc(M)) = End_{R/A}(Soc(M))$. Also $End_R(Soc(M)) \cong \bigoplus_{i=1}^t End_R(H_i)$ and each $End_R(H_i)$ is finite dimensional over C_i. Therefore $End_R(Soc(M))$ is a finitely generated C-module.

For more description on $End_R(Soc(M))$, for each $i = 1, 2, \ldots, t$, say $H_i = U_1 \oplus U_2 \oplus \cdots \oplus U_i$ (k_i-times) with U_i a simple R-module. Then we have that $End_R(H_i) = End_{R/P_i}(H_i) = Mat_{k_i}(End_{R/P_i}(U_i))$. Since $R/P_i = Mat_{n_i}(D_i)$, for some positive integer n_i and a division ring D_i, $End_{R/P_i}(H_i) = Mat_{m_i}(D_i)$, where $m_i = k_i l_i$. Also in this case $D_i = End_{R/P_i}(U_i)$ and $Z(D_i) = C_i$. Now note that $End_R(Soc(M)) = End_{R/A}(Soc(M)) = \bigoplus_{i=1}^t End_{R/A}(H_i)$. Therefore we have that $Z(End_R(Soc(M)) = C_1 \oplus C_2 \oplus \cdots \oplus C_t = Z(R/A)$.

On the other hand, since $Soc(M)$ is invariant under every endomorphism in $End_R(M)$, the map which associate with each $f \in End_R(M)$ its restriction to $Soc(M)$ becomes a ring homomorphism from $End_R(M)$ to $End_(Soc(M))$ with the kernel $N(S)$. In particular, for every $a \in Z(R)$, the right multiplication a_r on M is in S and so $a_r + N(S)$ in $S/N(S)$ associates with the restriction of a_r to $Soc(M)$ in $End_{R/A}$ $(Soc(M))$. Hence the subring $(Z(R) + N(S))/N(S)$ of $S/N(S)$ can be embedded as a central subring of $End_{R/A}(Soc(M))$. Thus $(Z(R) + N(S))/N(S) \subseteq Z(R/A)$ and so we may identify $a_r + N(S)$ with $a + A$ as an element in R/A.

For our convenience, let $B = S/N(S)$. Then since $BZ(R/A)$ is a $Z(R/A)$–submodule of $End_{R/A}(Soc(M))$ and $Z(R/A)$ is a finite direct sum of fields, $BZ(R/A)$ is a finitely generated $Z(R/A)$–module. Hence $BZ(R/A)$ is an Artinian ring and by [S] $BZ(R/A)$ is integral over $Z(R/A)$. Since R is integral over $Z(R)$, it can be easily checked that $Z(R/A)$ is integral over its subring $(Z(R) + N(S))/N(S)$, and consequently $BZ(R/A)$ is integral over B.

Moreover, $BZ(R/A)$ is an extension of B. Indeed, let C be the centralizer of B in the ring $BZ(R/A)$. Then $Z(R/A) \subseteq C$ and so $BZ(R/A) \subseteq BC \subseteq BZ(R/A)$. Therefore $BZ(R/A) = BC$ and hence $BZ(R/A)$ is a PI–ring which is an integral extension of B. So by Schelter [S, Theorem 1], GU (Going-Up) and LO (Lying-Over) hold between B and $BZ(R/A)$. As we already observed, since $BZ(R/A)$ is right Artinian, it has only finitely many maximal ideals. Thus B also has only finitely many maximal ideals. Since B is a PI–ring, B is semilocal. Finally since $N(S) \subseteq J(S)$ and $B = S/N(S)$ is semilocal, $S/J(S)$ is semilocal. So the ring $S/J(S)$ is Artinian and hence S is a semilocal ring.

Additionally, we have $J(BZ(R/A)) \cap B = J(B)$ and so $J(B)$ is nilpotent. Thus $S/N(S)$ is semiprimary and $J(S)^k \subseteq N(S)$ for some positive integer k.

Case II. Assume that R is an affine PI-algebra over a commutative Jacobson ring K. By using same notations and methods as in the proof of Case I, the subring $(K + N(S))/N(S)$ of $S/N(S)$ can be identified with the central subring $(K + A)/A$ of $End_R(Soc(M))$. Since R is affine over K, R/A is affine over $(K + A)/A$ and so it is affine over $(K + N(S))/N(S)$. Therefore for every central idempotent e of R/A,

$e(R/A)e$ is affine over $e((K+N(S))/N(S))e$. Particularly, R/P_i is affine over $e_i((K+N(S))/N(S))e_i = (K+P_i)/P_i$, where e_i is the block idempotent of R/A such that $e_i(R/A)e_i = R/P_i$.

Note R/P_i is finite dimensional over $Z(R/P_i)$. So by Artin-Tate lemma, there is a $(K+P_i)/P_i$-subalgebra L of $Z(R/P_i)$ such that R/P_i is a finitely generated L-module and L is affine over $(K+P_i)/P_i$. Now R/P_i is a finite centralizing extension of L and so the domain L is Artinian by Lemonnier [L]. Thus L is a field and so R/P_i is finite dimensional over the field L.

Since the field L is affine over $(K+P_i)/P_i$, $(K+P_i)/P_i (= K/(K\cap P_i))$ is a G-domain. So the prime ideal $K\cap P_i$ of the Jacobson ring K is a G-ideal and hence it is a maximal ideal of K. Thus $(K+P_i)/P_i$ is a field and so L is finite dimensional over $(K+P_i)/P_i$. By this fact, $R/P_i = Mat_{n_i}(D_i)$ is finite dimensional over $(K+P_i)/P_i$. In particular, D_i is finite dimensional over $(K+P_i)/P_i$.

On the other hand, recall that $K\cap P_i$ is a maximal ideal of K. Also $K\cap A = (K\cap P_1)\cap(K\cap P_2)\cap\cdots\cap(K\cap P_t)$. Thus by Chinese remainder theorem, $K/(K\cap A)\cong K/(K\cap P_1)\oplus K/(K\cap P_2)\oplus\cdots\oplus K/(K\cap P_t)$ and so $(K+A)/A \cong (K+P_1)/P_1\oplus(K+P_2)/P_2\oplus\cdots\oplus(K+P_t)/P_t$. But since $(K+N(S))/N(S) = (K+A)/A$, we have that $(K+N(S))/N(S)\cong (K+P_1)/P_1\oplus(K+P_2)/P_2\oplus\cdots\oplus(K+P_t)/P_t$.

Now since each D_i is finite dimensional over $(K+P_i)/P_i$, $End_R(Soc(M))$ is a finitely generated module over the semisimple Artinian ring $(K+N(S))/N(S)$. By noting that $(K+N(S))/N(S)$ is a subring of $S/N(S)$, the ring $S/N(S)$ is finitely generated as a module over $(K+N(S))/N(S)$. Therefore $S/N(S)$ is an Artinian ring and so the ring S is a semilocal ring.

From Theorem 1, we get the following interesting

Corollary 2. *Asssume that a ring R is either finitely generated as a module over its center or an affine PI-algebra over a field. Then the endomorphism ring of an Artinian R-module is semilocal.*

Recall that a ring R has *stable range* 1 provided that whenever $ax+b=1$ in R, there exists c in R such that $a+bc$ is a unit in R. By a result of Evans [E], stable range in endomorphism rings implies cancellation in direct sums; that is, if A and B are R-modules such that $M \oplus A \cong M \oplus B$ and $End_R(M)$ has stable range 1, then $A \cong B$.

From Theorem 1, we also can get the following interesting fact which can be compared with Theorem 2.8 in [CM].

Corollary 3. *Assume that R is a PI–ring which is either integral over its center or an affine algebra over a commutative Jacobson ring. Then the endomorphism ring of an Artinian R-module has stable range 1.*

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