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CHARACTRIZATION OF EXTREME $GTT - (0, \frac{1}{2}, 1)-$ MATRICES

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1. Introduction and basic definitions

A tournament matrix of order n is a $(0,1)$-matrix $M = [m_{ij}]$ which satisfies

(1) $\quad m_{ii} = 0,\ (i = 1, ..., n)$ and $\quad m_{ij} + m_{ji} = 1 \ (i \neq j)$.

The tournament matrix M is *transitive*, say a TT-matrix, provided it also satisfies

(2) $\quad m_{ij} + m_{jk} + m_{ki} \geq 1 \ (i, j, k$ distinct).

A generalized tournament matrix, GT-matrix, of order n is a nonnegative matrix M which satisfies (1). A generalized transitive tournament matrix, is a generalized tournament matrix satisfing (2). The set of all GT-matrices of order n forms a convex polytope $\mathcal{G}_n$ whose extreme points are the tournament matrices of order n. The set of all GTT-matrices of order n also forms a convex polytope $\mathcal{T}_n$, while the TT-matrices are extreme points of $\mathcal{T}_n$ there are in general other extreme points. We say that a GTT -matrix is *extreme* provided it is an extreme point of $\mathcal{T}_n$. Let $\mathcal{T}_n^*$ denote the convex hull of the TT-matrices of order n. It is known that $\mathcal{T}_n = \mathcal{T}_n^*$ only for $n \leq 5$ [4,6]. For $n \geq 6$, there is no known charactrazation of $\mathcal{T}_n$ by a finite set of linear constraints. For each GT- matrix M, we associate a graph($*$ -graph of M) whose edges correspond to the non-integral entries of M. A graph G is *GTT-realizable* provided there exists a GTT-matrix whose $*$-graph is isomorphic to G. A graph G is *transitively orientable* provided it is possible to orient each edge of G so that the resulting digraph satisfies the transitive law:

$$a \to b,\ b \to c \text{ implies } a \to c.$$

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A graph with transitive orientation is called a *comparability graph.*

2. Preliminaries

Theorem 1 *A graph G is GTT-realizable if and only if the compliment $\bar{G}$ is a comparability graph.*

Proof Let $M = [m_{ij}]$ be a GTT-matrix of order n and let G denote its $*$-graph. Choosing for each edge $\{i, j\}$ of $\bar{G}$ the orientation $i \to j$ if $m_{ij} = 1$ we obtain a transitive orientation of $\bar{G}$. Conversely, suppose $\bar{G}$ has a transitive orientation. We define a GT-matrix $M = [m_{ij}]$ by:

$$m_{ij} = \begin{cases} \frac{1}{2} \text{ if } \{i,j\} \text{ is an edge of } G, \\ 1 \text{ if } \{i,j\} \text{ is an edge of } \bar{G} \text{ with orientation } i \to j, \\ 0 \text{ otherwise.} \end{cases}$$

If $m_{ij} = m_{jk} = 1$, then the transitive orientation of $\bar{G}$ implies that $\{i, k\}$ is an edge of $\bar{G}$ and $m_{ik} = 1$. It now follows that M is a GTT-matrix with $*$-graph equal to G.

Comparability graphs have been charactrized by Gillmore and Hoffman[3](see Theorem 3), so we get the charactrization of GTT-realizable graphs by appling to the $\bar{G}$.

Let G be a graph with edge set E. Let

$$\hat{E} = \{ \ (a, b), (b, a) | (a, b) \in E \ \}.$$

Define binary relation Γ on $\hat{E}$ as follows.

$$(a, b) \ \Gamma \ (a', b') \text{ iff } \begin{cases} \text{either } a = a' \text{ and } \{b, b'\} \notin E \\ \text{or } b = b', \text{ and } \{a, a'\} \notin E. \end{cases}$$

The reflexive transitive closure Γ^* of Γ is an equivalence relation on $\hat{E}$ and equivalence class is called *implication class* of G. For each implication class I, define

$$I^{-1} = \{(a, b) : (b, a) \in I\}.$$

Lemma 2 *Let I be a implication class of a graph G. Exactly one of the following holds;*

i) $I \cap I^{-1} = \emptyset$

ii) $I = I^{-1}$.

Proof Assume $I \cap I^{-1} \neq \emptyset$. Let $(a,b) \in I \cap I^{-1}$, so $(a,b)\Gamma^*(b,a)$. For any $(c,d) \in I$, $(c,d)\Gamma^*(a,b)$ and $(d,c)\Gamma^*(b,a)$. Since Γ^* is an equivalence relation, $(c,d)\Gamma^*(d,c)$ and $(d,c) \in I$. Thus $I = I^{-1}$.

Theorm 3 *Let G be a undirected graph with edge set E. $\hat{E}$ is defined as above. The following statements are equivalence:*

i) G is a comparability graph.

ii) $I \cap I^{-1} = \emptyset$ for all implication classes I of $\hat{E}$.

iii) Every circuit of edges $\{a_1,a_2\},\{a_2,a_3\},\cdots,\{a_n,a_1\} \in E$ such that $\{a_{n-1},a_1\},\{a_n,a_2\},\cdots,\{a_{i-1},a_{i+1}\} \notin E$ $(i = 2,\cdots,n-1)$ has even length.

3. Main Results

Theorem 4 *Let $M = [m_{ij}]$ be a GTT-matrix whose $*$-graph is a G with at least one edge. M is a extreme $GTT - (0,\frac{1}{2},1)$- matrix if and only if $I \cap I^{-1} \neq \emptyset$ for all implication classes I.*

Proof Suppose that $I \cap I^{-1} = \emptyset$ for some implicatiion class I of G. Let $M(\epsilon) = [m_{ij}(\epsilon)]$ be obtained from M by adding ϵ to m_{ij} if $(i,j) \in I$ and subtracting ϵ from m_{ji} if $(j,i) \in I^{-1}$. We claim that $M(\epsilon)$ is a GTT- matrix. Because $I \cap I^{-1} = \emptyset, M(\epsilon)$ is a GT-matrix. It suffices that $M(\epsilon)$ satisfies

$$(4) \quad 1 \leq m_{ij}(\epsilon) + m_{jk}(\epsilon) + m_{ki}(\epsilon) \leq 2$$

for all distinct i,j,k. If none of $(i,j),(j,k)$ and (k,i) is in I then $M(\epsilon)$ satisfies transitive inequality (4). Assume that at least one of $(i,j),(j,k)$ and (k,i),say (i,j) is in I and thus m_{ij} is strictly between 0 and 1. One of the following holds:

(i) Both of m_{jk} and m_{ki} are integers (0 or 1). Since M is a GTT-matrix, both of them can not be 0(or 1) and thus $1 < m_{ij}+m_{jk}+m_{ki} < 2$.

(ii) Only one of m_{jk} and m_{ki}, say m_{jk}, is an integer. Since $(i,j) \in I$ and $\{j,k\}$ is not an edge of G, we have $(i,k) \in I$. $I \cap I^{-1} = \emptyset$ implies $(k,i) \in I^{-1}$.

(iii) Neither m_{jk} nor m_{ki} is integer. Since $m_{ij}=m_{jk}=m_{ki}=\frac{1}{2}$, $1<m_{ij}+m_{jk}+m_{ki}<2$.

It follows from (i),(ii) and (iii) that for ϵ a small positive number, $M(\epsilon)$ satisfies (4). By same argument $M(-\epsilon)$ is a GTT- matrix. We have

$$(5)\quad M=\tfrac{1}{2}(M(\epsilon)+M(-\epsilon)),$$

so M is not extreme.

Conversely, suppose that $I\cap I^{-1}\neq\emptyset$ for all implication class I of G and let $M=\frac{1}{2}(A+B)$ for some GTT-matrices $A=[a_{ij}]$ and $B=[b_{ij}]$. We have

$$m_{ij}=a_{ij}=b_{ij}(=0 \text{ or } 1) \text{ if } \{i,j\} \text{is not an edge of } G$$
$$m_{ij}=\frac{1}{2}=\frac{1}{2}(a_{ij}+b_{ij}),\ b_{ij}=1-a_{ij} \text{ if } \{i,j\} \text{ is an edge of } G.$$

Suppose that $(i,j)\ \Gamma\ (i,k)$ for some distinct i,j,k. Then $\{j,k\}$ is not an edge of G, and so $m_{jk}=a_{jk}=b_{jk}$ is an integer (0 or 1). Without loss of generality assume that $m_{jk}=1$. Then we have $a_{ij}+a_{ki}\leq 1$ and $b_{ij}+b_{ki}\leq 1$. If $a_{ij}+a_{ki}<1$ then $b_{ij}+b_{ki}=2-(a_{ij}+a_{ki})>1$, contradicting the fact that B is GTT-matrix. Hence $a_{ij}+a_{ki}=1$, and so $a_{ij}=1-a_{ki}=a_{ik}$. Therfore $a_{ij}=a_{ik}$ whenever $(i,j)\ \Gamma\ (i,k)$. Now

$$(i,j)\ \Gamma^*\ (i',j') \text{ iff}$$
$$\exists\ \Gamma-\text{chain } (i,j)=(i_1,j_1)\Gamma(i_2,j_2)\Gamma\cdots\Gamma(i_k,j_k)=(i',j').$$

Hence we have $a_{ij}=a_{i'j'}$ if $(i,j)\ \Gamma^*\ (i',j')$. Let $(i,j)\in I$ for some implication class I. Then $a_{ij}=a_{i'j'}$ for all $(i',j')\in I$. $I\cap I^{-1}\neq\emptyset$ means $I=I^{-1}$, so $a_{ij}=a_{ji}$. Thus $a_{ij}=\frac{1}{2}=b_{ij}$ for $(i,j)\in I$. By the same argument, $a_{ij}=b_{ij}=\frac{1}{2}$ for all edges $\{i,j\}$ of G. Hence $M=A=B$, so M is an extreme $GTT-(0,\frac{1}{2},1)$- matrix.

Corollary 5 *The $*$-graph of any extreme $GTT-(0,\frac{1}{2},1)-matrix$ with at least one edge is not a comparability graph, but its compliment is a comparability graph.*

We now see the relation between extreme $GTT-(0,\frac{1}{2},1)$- matrices and examples of GTT-nonrealizable graphs.

Lemma 6 *If the compliment $\bar{G}$ of a graph G is a even-cycle C_n for $n \geq 6$, then G is $*$-graph of an extreme $GTT - (0, \frac{1}{2}, 1)$- matrix.*

Proof Suppose that $\bar{G}$ is even-cycle $C_n = (1, 2, \cdots, n, 1), n \geq 6$. Since an even-cycle is a comparability graph, G is GTT-realizable graph. Let $M = [m_{ij}]$ be a GTT-matrix whose $*$-graph is G and $m_{ij} = \frac{1}{2}$ if $\{i, j\}$ is an adge of G. Assume that $M = \frac{1}{2}(A + B)$ for some GTT- matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Then $m_{ij} = a_{ij} = b_{ij}$ are integers if $\{i, j\}$ is an adge of $\bar{G}$. We get, after reordering if necessary,

$$A = \begin{bmatrix} 0 & * & & & & * \\ * & 0 & * & & \beta & \\ & & \cdots & & & \\ & \bar{\beta} & & * & 0 & * \\ * & & & & * & 0 \end{bmatrix}$$

where $*$ is 0 or 1 and $0 \leq \beta \leq 1$, $\bar{\beta} = 1 - \beta$. Then $a_{13} = a_{3n} = \beta$ and a_{n1} equals 0 or 1. This implies that $a_{13} + a_{3n} = 2\beta = 1$, thus $M = A = B$. Therefore M is an extreme $GTT - (0, \frac{1}{2}, 1)$- matrix.

Lemma 7 *If the compliment $\bar{G}$ of a graph G contains a chordless $k-$cycle($k \geq 5$, odd) as induced subgraph, then G is GTT-nonrealizable graph.*

Proof Suppose that $\bar{G}$ contains a cycle $C_k = (1, 2, \cdots, k, 1)$. Assume that $M = [m_{ij}]$ is the GTT-matrix whose $*$-graph is G and $m_{ij} = \frac{1}{2}$ for all i and j such that $\{i, j\}$ is an adge of G. Let $P = [p_{ij}]$ be the principle submatrix of M of order k, whose $*$-graph is compliment of C_k. Then p_{ij} are integers if $\{i, j\}$ is an adge of C_k. Whitout loss of generality, assume that $p_{12} = 1$ (the possibility $p_{12} = 0$ is argued in a similar way). Since P is also GTT-matrix we have $p_{ii+1} = 1$ for all odd i, $(i = 1, 3, \cdots, k-2)$ and $p_{1k} = 1$. Thus $p_{1k} + P_{k-1k} + p_{k1} = \frac{1}{2} < 1$, contradicting the transitivity of P. Hence G is GTT-nonrealizable graph.

Lemma 8 *Let a graph G be the compliment of LB_n in figure 1 for $n \geq 6$. Then G is GTT-realizable graph if n is even, and G is $*$ graph of extreme $GTT - (0, \frac{1}{2}, 1)$- matrix if n is odd.*

Proof Let $\bar{G}$ be an LB_n in Figure 1. Assume that n is even and $M = [m_{ij}]$ is a GTT-matrix whose $*$-graph is G and $m_{ij} = \frac{1}{2}$

for each edge $\{i,j\}$ of G. Without loss of generality, assume that $m_{12}=1$. Repeated use of the transitive inequality gives that $m_{1k}=1$ for $k=3,\cdots,n-3,n-1$ and $m_{kk+1}=0$ for even k, $k\leq n-3$ and $m_{kk+1}=1$ for odd k, $k\leq n-3$. Hence $m_{1n-3}=m_{n-3n-2}=1$ and $m_{n-21}=\frac{1}{2}$, contradicting the transitivity of M.

Now assume that n is odd. Let $M=[m_{ij}]$ be a GT-matrix such that $m_{1k}=1$ for $k=2,\cdots,n-3,n-1$ and $m_{kk+1}=1$ for odd k where $k\leq n-3$, otherwise $m_{ij}=\frac{1}{2}$. Then it is easy to check that M satisfies transitive inequality. Assume that $M=\frac{1}{2}(A+B)$ for some GTT-matrices A and B. We get $M=A=B$ by the same argument in Lemma 6. Hence M is an extreme $GTT-(0,\frac{1}{2},1)-$ mareix.

Theorem 9. *The compliment of $*$-graph of any extreme $GTT-(0,\frac{1}{2},1)$- matrix of order 6 is isomorphic to C_6 or G_1 in figure 1. Therefore M_1 and M_2 are the only extreme $GTT-(0,\frac{1}{2},1)$- matrices of order 6 up to isomorphism.*

Proof. If G is a $*$-graph of an extreme $GTT-(0,\frac{1}{2},1)$- matrix, then $\bar{G}$ is GTT-nonrealizable. Note that C_5, C_6, LB_6 and two graphs G_1, G_2 are the only minimal GTT-nonrealizable graphs of order at most 6. If $\bar{G}$ contains C_5 or $\bar{G}=LB_6$, then G is GTT- nonrealizable by Lemma 7 and 8. Hence the only possible compliments of $*$-graph of any extreme $GTT-(0,\frac{1}{2},1)$- matrix of order 6 are C_6 and G_1, and these two are compliment of $*$-graphs of M_1 and M_2.

$$M_1=\begin{pmatrix} 0 & \frac{1}{2} & 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \quad M_2=\begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 & \frac{1}{2} & 1 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

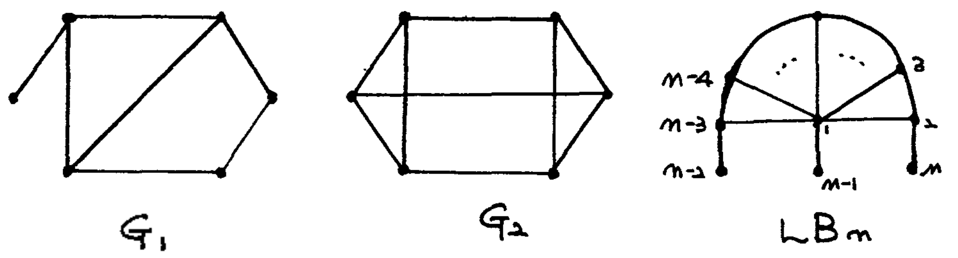

Figure 1

Referance

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