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# BETWEEN FUZZY STRONG Θ–CONTINUITY AND FUZZY WEAK Θ–CONTINUITY

Bu Young Lee, Jin Han Park and Oug Kim

## 1. Introduction and preliminaries

Weaker and stronger forms of fuzzy continuous mappings on fuzzy topological spaces have been considered by many authors [1,3,4,7,8,11] by using the concept of q-coincidence. Rencently, J.H. Park et al. [11] studied some characterizing theorems for fuzzy strong $\theta$-continuity, fuzzy $\theta$-continuity and fuzzy ~~weakly~~ $\theta$-continuity in fuzzy topological spaces by using the concepts of fuzzy $\delta$-closure [4] and fuzzy $\theta$-closure [8].

In this paper, we study some properties of fuzzy strong $\theta$-continuous, fuzzy almost strong $\theta$-continuous and fuzzy weakly $\theta$-continuous mappings by using the concepts of fuzzy $\delta$-interior and fuzzy $\theta$-interior, and invetigate the relationships among them under suitably conditions.

Throughout this paper, by $(X,\tau)$ (or simply $X$) we mean a fuzzy topological space in Chang's [2] sense. A fuzzy point in $X$ with support $x \in X$ and value $\alpha(0 < \alpha \leq 1)$ is denoted by $x_\alpha$. For a fuzzy set $A$ in $X$, Cl$A$, Int$A$ and $1 - A$ will respectively denote the closure, interior and complement of $A$, whereas the constant fuzzy sets taking on the values 0 and 1 on $X$ are denoted by $0_X$ and $1_X$, respectively. A fuzzy set $A$ of $X$ ia said to be q-coincident with a fuzzy set $B$, denoted by $A\text{q}B$, if there exists $x \in X$ such that $A(x) + B(x) > 1$[6]. It is known [6] that $A \leq B$ iff $A$ and $1 - B$ are not q-coincident, denoted by $A\tilde{\text{q}}(1 - B)$. For definitions and results not explained in this paper, the reader is referred to [1,2,6] in the assumption they are well known. The words 'fuzzy', 'neighborhood' and 'fuzzy topological space' will be abbreviated as 'f.', 'nbd' and 'fts', respectively.

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DEFINITION 1.1 [4, 8]. A f.point $x_\alpha$ is said to be a f.$\delta$-cluster point ($\theta$-cluster point) of a f.set $A$ in $X$ iff f.interior of the closure (resp. closure) of every f.open q-nbd $U$ of $x_\alpha$ is q-coincident with $A$. The union of all f.$\delta$-cluster ($\theta$-cluster) points of $A$ ia called the f.$\delta$-closure (resp. $\theta$-closure) and is denoted as $\mathrm{Cl}_\delta A$ (resp. $\mathrm{Cl}_\theta A$). $A$ ia called f.$\delta$-closed ($\theta$-closed) iff $A = \mathrm{Cl}_\delta A$ (resp. $A = \mathrm{Cl}_\theta A$) and the complement of a f.$\delta$-closed (resp. $\theta$-closed) set is f.$\delta$-open (resp. $\theta$-open).

LEMMA 1.1 [9, 12]. *Let $A$ be a f.set of a fts $X$.*
*(a) $Cl_\delta A = \cap\{B \mid B$ is f.regularly closed and $A \le B\}$.*
*(b) $Cl_\theta A = \cap\{ClB \mid B$ is f.open and $A \le B\}$.*

DEFINITION 1.2. Let $A$ be any f.set of a fts $X$. Then f.$\delta$-interior ($\mathrm{Int}_\delta$) and f.$\theta$-interior ($\mathrm{Int}_\theta$) of $A$ are defined as follows:

$$\mathrm{Int}_\delta A = \cup\{B \mid B \text{ is f.regularly open and } B \le A\},$$
$$\mathrm{Int}_\theta A = \cup\{\mathrm{Int}B \mid B \text{ is f.closed and } B \le A\}.$$

LEMMA 1.2. *For any f.set $A$ in a fts $X$, $1 - Int_\delta A = Cl_\delta(1 - A)$ and $1 - Int_\theta A = Cl_\theta(1 - A)$.*

## 2. Chracterizations

DEFINITION 2.1 [7, 8]. A mapping $f : X \to Y$ is said to be f.strong $\theta$-continuous (f.almost strong $\theta$-continuous, f.weakly $\theta$-continuous) if for each f.point $x_\alpha$ in $X$ and each f.open q-nbd $V$ of $f(x_\alpha)$, there exists f.open q-nbd $U$ of $x_\alpha$ such that $f(\mathrm{Cl}U) \le V$ (resp. $f(\mathrm{Cl}U) \le \mathrm{IntCl}V$, $f(\mathrm{IntCl}U) \le \mathrm{Cl}V$).

THEOREM 2.1. *A mapping $f : X \to Y$ is f.strong $\theta$-continuous iff $f^{-1}(IntB) \le Int_\theta f^{-1}(B)$ for each f.set $B$ in $Y$.*

*Proof.* Let $B$ be a f.set in $Y$. Then by Theorem 2.2 in [11] and Lemma 1.2, we have $f^{-1}(\mathrm{Int}B) = 1 - f^{-1}(\mathrm{Cl}(1-B)) \le 1 - \mathrm{Cl}_\theta f^{-1}(1-B) = \mathrm{Int}_\theta f^{-1}(B)$.

Conversely, let $B$ be a fuzzy open in $Y$. By hypothesis, we have $f^{-1}(B) = f^{-1}(\mathrm{Int}B) \le \mathrm{Int}_\theta f^{-1}(B)$ which implies $f^{-1}(B) = \mathrm{Int}_\theta f^{-1}(B)$. Then $f^{-1}(B)$ is f.$\theta$-open and hence $f$ is f.strong $\theta$-continuous from Theorem 2.2 in [11].

THEOREM 2.2. *For a mapping $f : X \to Y$, the following are equivalent:*

*(a) $f$ is f.almost strong $\theta$-continuous.*

*(b) $f(Cl_\theta A) \leq Cl_\delta f(A)$ for each f.set $A$ in $X$.*

*(c) $f^{-1}(Int_\delta B) \leq Int_\theta f^{-1}(B)$ for each f.set $B$ in $Y$.*

*Proof.* (a)⇒(b): Let $x_\alpha \in \mathrm{Cl}_\theta A$ and $V$ be a f.open q-nbd of $f(x_\alpha)$. Then there exists a f.open q-nbd $U$ of $x_\alpha$ such that $f(\mathrm{Cl}U) \leq \mathrm{IntCl}V$. Now we have

$$x_\alpha \in \mathrm{Cl}_\theta A \Rightarrow \mathrm{Cl}U \mathrm{q} A \Rightarrow f(\mathrm{Cl}U)\mathrm{q}f(A) \Rightarrow \mathrm{IntCl}V\mathrm{q}f(A)$$
$$\Rightarrow f(x_\alpha) \in \mathrm{Cl}_\delta f(A) \Rightarrow x_\alpha \in f^{-1}(\mathrm{Cl}_\delta f(A)).$$

Hence $\mathrm{Cl}_\theta A \leq f^{-1}(\mathrm{Cl}_\delta f(A))$ and so $f(\mathrm{Cl}_\theta A) \leq \mathrm{Cl}_\delta f(A)$.

(b)⇒(c): Taking the complement implies the proof.

(c)⇒(a): It is similar to the proof of Theorem 2.1 by using Theorem 2.12 in [8].

THEOREM 2.3. *For a mapping $f : X \to Y$, the following are equivalent:*

*(a) $f$ is f.weakly $\theta$-continuous.*

*(b) $f^{-1}(B) \leq Int_\delta f^{-1}(ClB)$ for each f.open set $B$ of $Y$.*

*(c) $Clf^{-1}(Int_\delta B) \leq f^{-1}(B)$ for each f.closed set $B$ of $Y$.*

*(d) $f^{-1}(Int_\theta B) \leq Int_\delta f^{-1}(B)$ for each f.set $B$ of $Y$.*

*Proof.* (a)⇒(b): It is clear from Theorem 2.5 in [11] and Lemma 1.2.

(b)⇒(a): Let $x_\alpha$ be a f.point in $X$ and $V$ be a f.open q-nbd of $f(x_\alpha)$. By (b), we have $x_\alpha \mathrm{q} f^{-1}(V) \leq \mathrm{Int}_\delta f^{-1}(\mathrm{Cl}V)$. Then by lemma 1.2 we have

$$x_\alpha \mathrm{q} \mathrm{Int}_\delta f^{-1}(\mathrm{Cl}V) \Rightarrow x_\alpha \notin 1 - \mathrm{Int}_\delta f^{-1}(\mathrm{Cl}V) = \mathrm{Cl}_\delta(1 - f^{-1}(\mathrm{Cl}V)).$$

Hence there exists a f.open q-nbd $U$ of $x_\alpha$ such that $\mathrm{IntCl}U\bar{\mathrm{q}}1 - f^{-1}(\mathrm{Cl}V)$ so that $\mathrm{IntCl}U \leq f^{-1}(\mathrm{Cl}V)$. This shows that $f$ is f.weakly $\theta$-continuous.

(b)⇔(c): Taking the complement implies the proof.

(a)⇔(d): Similar to the proof of Theorem 2.1.

THEOREM 2.4. *Let the mapping $f : X \to Y$ f.strong $\theta$-continuous and $g : Y \to Z$ be f.continuous. Then the composite mapping $g \circ f$ is f.strong $\theta$-continuous.*

*Proof.* Let $V$ be a f.open set in $Z$. Then $g^{-1}(V)$ is f.open set in $Y$ so that $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is f.$\theta$-open set in $X$ by Theorem 2.1 in [11]. Thus $g \circ f$ is f.strong $\theta$-continuous.

COROLLARY 2.5. *Let the mapping $f : X \to Y$ f.strong $\theta$-continuous and $g : Y \to Z$ be f.strong $\theta$-continuous. Then the composite mapping $g \circ f$ is f.strong $\theta$-continuous.*

THEOREM 2.6. *Let $X$ and $Y$ be fts's such that $X$ is product related to $Y$. Let $f : X \to Y$ be a mapping and $g : X \to X \times Y$, given by $g(x) = (x, f(x))$ for each $x$ of $X$, be the graph mapping. Then the following are true:*

*(a) $f$ is f.weakly $\theta$-continuous. if and only if $g$ is f.weakly $\theta$-continuous.*

*(b) If $g$ is f.strong $\theta$-continuous, then $f$ is f.strong $\theta$-continuous.*

*Proof.* (a): Let $x_\alpha$ be a f.point in $X$ and $V$ be any f.open q-nbd of $f(x_\alpha)$. Then $1_X \times V$ is f.open q-nbd of $g(x_\alpha)$. Since $g$ is f.wealky $\theta$-continuous, there exists a f.open q-nbd $U$ of $x_\alpha$ such that $g(\text{IntCl}U) \leq \text{Cl}(1_X \times V) = 1_X \times \text{Cl}V$. By Lemma 2.9 in [7], we have $f(\text{IntCl}U) \leq \text{Cl}V$. Hence $f$ is f.weakly $\theta$-continuous.

Conversely, let $x_\alpha$ be a f.point in $X$ and $W$ be any f.open q-nbd of $g(x_\alpha)$ in $X \times Y$. By Lemma 2.9 in [7], there exist f.open q-nbd $U$ of $x_\alpha$ and f.open q-nbd $V$ of $f(x_\alpha)$ such that $g(x_\alpha)\text{q}(U \times V) \leq W$. Since $f$ is f.weakly $\theta$-continuous, there exists f.open q-nbd $G$ of $x_\alpha$ such that $G \leq U$ and $f(\text{IntCl}G) \leq \text{Cl}V$. Now we have

$$\begin{aligned} g(\text{IntCl}G) &= \text{IntCl}G \times f(\text{IntCl}G) \\ &\leq \text{Cl}U \times \text{Cl}V = \text{Cl}(U \times V) \\ &\leq \text{Cl}W. \end{aligned}$$

Hence $g$ is f.weakly $\theta$-continuous.

(b): Similar to the proof of (a).

DEFINITION 2.2. A mapping $f : X \to Y$ is said to be f.weakly continuous [7] (f.$\theta$-continuous [8], f.almost continuous [8], f.$\delta$-continuous [4], f.super continuous [7]) if for each f.point $x_\alpha$ in $X$ and each f.open q-nbd $V$ of $f(x_\alpha)$, there exists f.open q-nbd $U$ of $x_\alpha$ such that $f(U) \leq \mathrm{Cl}V$ (resp. $f(\mathrm{Cl}U) \leq \mathrm{Cl}V$, $f(U) \leq \mathrm{IntCl}V$, $f(\mathrm{IntCl}U) \leq \mathrm{IntCl}V$, $f(\mathrm{IntCl}U) \leq V$).

It is clear that f.strong $\theta$-continuity implies f.almost strong $\theta$-conti nuity and f.super continuity, and f.super continuity implies f.continuity. But f.almost strong $\theta$-continuity need not be f.continuity (see [8]). From following example and Examlpe 3.10 in [8], we know that f.almost strong $\theta$-continuity and f.super continuity (f.continuity) are independent concepts.

EXAMPLE. Let $X = [0,1]$ and $\tau = \{1_X, 0_X, A\}$, where $A(0) = \frac{1}{3}$ and $A(x) = 0$ for $x \neq 0$.

Consider the identy mapping $f : (X,\tau) \to (X,\tau)$. We show that $f$ is f.super continuous but not f.almost strong $\theta$-continuous. Let $x_\alpha$ be a f.point in $X$. If $x \neq 0$, then $V = 1_X$ is the only f.open q-nbd of $f(x_\alpha)$, and then $U = 1_X$ is a f.open q-nbd of $x_\alpha$ such that $f(\mathrm{IntCl}U) \leq V$. Suppose $x = 0$ and $V$ is f.open q-nbd of $f(x_\alpha)$. If $V = 1_X$, the case becomes trivial. So let $V = A$. Then $\alpha > \frac{2}{3}$ so that $A$ is a f.open q-nbd of $x_\alpha$ such that $f(\mathrm{IntCl}A) = A$. Hence $f$ is f.super continuous.

Now consider the f.point $x_\alpha$, where $x = 0$ and $\alpha = \frac{5}{6}$. Then $A$ is a f.open q-nbd of $f(x_\alpha)$. Let $U$ be any f.open q-nbd of $x_\alpha$. Then $U = A$ or $1_X$, and $f(\mathrm{Cl}U) = 1 - A$ or $1_X \not\leq \mathrm{IntCl}A = A$. Hence $f$ is not f.almost strong $\theta$-continuous.

DEFINITION 2.3 [8]. A fts $X$ is said to be

(a) f.regular if for each f.point $x_\alpha$ in $X$ and each f.open q-nbd $U$ of $x_\alpha$, there exists a f.open q-nbd $V$ of $x_\alpha$ such that $\mathrm{Cl}V \leq U$.

(b) f.almost regular if for each f.regularly open set $V$ in $X$ and each f.point $x_\alpha qV$, there exists a f.regularly open set $U$ such that $x_\alpha qU \leq \mathrm{Cl}U \leq V$.

(c) f.semi-regular if for each f.open set $V$ in $X$ and each f.point $x_\alpha qV$, there exists a f.open set $U$ such that $x_\alpha qU \leq \mathrm{IntCl}U \leq V$.

It is easy to see that every f.regular space is f.semi-regular as well as f.almost regular [8].

THEOREM 2.7. *Let $X$ and $Y$ be fts's such that $X$ is product related to $Y$ and $g$ be the graph mapping of $f : X \to Y$. If $g$ is f.strong $\theta$-continuous, then $X$ is f.regular.*

*Proof.* Let $f$ be f.strong $\theta$-continuous and $x_\alpha$ be any f.point in $X$. Then for f.open q-nbd $V$ of $x_\alpha$, $V \times 1_Y$ is f.open q-nbd of $g(x_\alpha)$. Since $g$ is f.strong $\theta$-continuous, there exists f.open q-nbd $U$ of $x_\alpha$ such that $g(\mathrm{Cl}U) \leq V \times 1_Y$. This implies that $x_\alpha \mathrm{q} U \leq \mathrm{Cl}U \leq V$. Hence $X$ is f.regular.

THEOREM 2.8. *If $Y$ is f.regular and $f : X \to Y$ is f.continuous mapping, then $f$ is f.strong $\theta$-continuous.*

*Proof.* Let $x_\alpha$ be a f.point in $X$ and $V$ be f.open q-nbd of $f(x_\alpha)$. Then there exists f.open set $W$ such that $f(x_\alpha)\mathrm{q}W \leq \mathrm{Cl}W \leq V$. Since $f$ is f.continuous, there exist f.open q-nbd $U$ such that $f(U) \leq W$. This implies that $f(\mathrm{Cl}U) \leq \mathrm{Cl}f(U) \leq \mathrm{Cl}W \leq V$. Hence $f$ is f.strong $\theta$-continuous.

THEOREM 2.9. *Let $f : X \to Y$ be a mapping. Then the following are true:*

*(a) If $f$ is fuzzy weakly continuous and $X$ is f.semi-regular, then $f$ is f.weakly $\theta$-continuous.*

*(b) If $f$ is f.almost strong $\theta$-continuous and $Y$ is f.semi-regular, then $f$ is f.strong $\theta$-continuous.*

THEOREM 2.10. *Let $f : X \to Y$ be a mapping. Then the following are true:*

*(a) If $f : X \to Y$ is f.weakly $\theta$-continuous and $X$ is f.almost regular, then $f$ is f.$\theta$-continuous.*

*(b) If $f : X \to Y$ is f.weakly continuous and $Y$ is f.almost regular, then $f$ is f.almost strong $\theta$-continuous.*

*Proof.* (a): Easy.

(b) Let $x_\alpha$ be a f.point in $X$ and $V$ ba any f.regularly open q-nbd of $f(x_\alpha)$. Then there exists f.regularly open q-nbd $W$ such that $\text{Cl}W \leq V$. Since $f$ is f.weakly continuous, there exists f.open q-nbd $U$ of $x_\alpha$ such that $f(U) \leq \text{Cl}W \leq V$. Thus $f$ is f.almost continuous. By Theorem 3.1 in [3] and Theorem 3.8 (b) in [8], $f$ is f.almost strong $\theta$-continuous.

COROLLARY 2.11 [8]. *If $X$ is f.regular and $f : X \to Y$ is f.weakly continuous, then $f$ is f.$\theta$-continuous.*

*Proof.* Clear from Theorem 2.9 and Theorem 2.10.

The following two theorems are easily proved and the proofs are omitted.

THEOREM 2.12. *Let $f : X \to Y$ and $g : Y \to Z$ be mappings. If $f$ and $g$ are satisfied with one of the following, then the composition $g \circ f$ is f.weakly $\theta$-continuous.*

*(a) $f$ is f.super continuous and $g$ is f.weakly continuous.*

*(b) $f$ is f.$\delta$-continuous and $g$ is f.weakly $\theta$-continuous.*

*(c) $f$ is f.weakly $\theta$-continuous and $g$ is f.$\theta$-continuous.*

THEOREM 2.13. *Let $f : X \to Y$ and $g : Y \to Z$ be mappings.*

*(a) If $f$ is f.weakly $\theta$-continuous and $g$ is f.almost strong $\theta$-continuous, then $g \circ f$ is f.$\delta$-continuous.*

*(b) If $f$ is f.weakly $\theta$-continuous and $g$ is f.strong $\theta$-continuous, then $g \circ f$ is f.super continuous.*

*(c) If $f$ is f.almost strongly $\theta$-continuous and $g$ is f.weakly $\theta$-continuous, then $g \circ f$ is f.$\theta$-continuous*

*(d) If $f$ is f.almost continuous and $g$ is f.weakly $\theta$-continuous, then $g \circ f$ is f.weakly continuous.*

DEFINITION 2.4 [10]. A mapping $f : X \to Y$ is said to be f.almost open if $f(U)$ is f.open in $Y$ for each f.regularly open set $U$ of $X$.

THEOREM 2.14. *If a mapping $f : X \to Y$ is f.weakly $\theta$-continuous and f.almost open, then $f$ is f.$\delta$-continuous.*

*Proof.* Let $x_\alpha$ be a f.point in $X$ and $V$ be any f.open q-nbd of $f(x_\alpha)$. Since $f$ is f.weakly $\theta$-continuous, there exists f.open q-nbd $U$ of $x_\alpha$ such that $f(\text{IntCl}U) \leq \text{Cl}V$. Since $f$ is f.almost open, $f(\text{IntCl}U)$ is f.open and hence $f(\text{IntCl}U) \leq \text{IntCl}V$. This shows that $f$ is f.$\delta$-continuous.

COROLLARY 2.15 [8]. *If a mapping $f : X \to Y$ is f.$\theta$-continuous and f.almost open, then $f$ is f.$\delta$-continuous.*

DEFINITION 2.5 [5]. A fts $X$ is said to be f.extremally disconnected if the closure of each f.open set of $X$ is f.open.

THEOREM 2.16. *Let $X$ be a f.extremally disconnected space and $f : X \to Y$ be a mapping.*
*(a) If $f$ is f.weakly $\theta$-continuous, then $f$ is f.$\theta$-continuous.*
*(b) If $f$ is f.$\delta$-continuous, then $f$ is f.almost strongly $\theta$-continuous.*
*(c) If $f$ is f.super continuous, then $f$ is f.strongly $\theta$-continuous.*

*Proof.* We prove only the case that $f$ is f.weakly $\theta$-continuous and then the proofs of the other are similar.

Let $x_\alpha$ be a f.point in $X$ and $V$ be any f.open q-nbd of $f(x_\alpha)$. Since $f$ is f.weakly $\theta$-continuous, there exists a f.open q-nbd $U$ of $x_\alpha$ such that $f(\text{IntCl}U) \leq \text{Cl}V$. Since $X$ is f.extremally disconnected, we have $\text{IntCl}U = \text{Cl}U$ and hence $f(\text{Cl}U) \leq \text{Cl}V$. This shows that $f$ is f.$\theta$-continuous.

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Department of Mathematics
Dong-A University
Pusan 604-714, Korea