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GEOMETRIC CHARACTERIZATIONS OF JOHN DISKS

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1. Introduction

We say that a domain D in $\overline{\mathbb{C}}$ is a K- *quasidisk* if it is the image of the unit disk $\mathbb{B}$ under a K-quasiconformal self mapping of $\overline{\mathbb{C}}$. Quasidisks have been extensively studied and can be characterized in many different ways [1], [2].

We say that a domain D in $\mathbb{C}$ is c- *uniform* if there is a constant $c \geq 1$ such that each two points z_1 and z_2 in D can be joined by an arc γ in D such that

$$\ell(\gamma) \leq c|z_1 - z_2|$$

and

$$\min(\ell(\gamma_1), \ell(\gamma_2)) \leq c\, d(z, \partial D) \tag{1.1}$$

for all $z \in \gamma$, where γ_1 and γ_2 are the components of $\gamma \setminus \{z\}$. We say D is *uniform* if it is c- uniform for some $c \geq 1$. A Jordan domain D in $\mathbb{C}$ is uniform if and only if it is a quasidisk [8].

A bounded domain D in $\mathbb{C}$ is said to be a *c-John domain* if there exist a point $z_0 \in D$ and a constant $c \geq 1$ such that each point $z_1 \in D$ can be joined to z_0 by an arc γ in D satisfying

$$\ell(\gamma(z_1, z)) \leq c\, d(z, \partial D)$$

for each $z \in \gamma$. We call z_0 a *John center*, c a *John constant* and γ a *c-John arc.*

There are several equivalent definitions for John domains. For example, a bounded domain D in $\mathbb{C}$ is a c-John domain if and only if each two points $z_1, z_2 \in D$ can be joined by an arc γ which satisfies

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(1.1). This definition can be used to define the unbounded John domains D in $\overline{\mathbb{C}}$ as well [10, 2.26]. Therefore the class of uniform domains is properly contained in the class of John domains [4], [6], [10].

We say that a domain $D \subset \overline{\mathbb{C}}$ is a *c-John disk* if it is a simply connected c-John domain.

Gehring and Osgood show in [6] that a domain D in $\mathbb{C}$ is uniform if and only if it is quasiconformally decomposable, i.e., for each $z_1, z_2 \in D$ there exists a K-quasidisk G_0 in D such that $z_1, z_2 \in \overline{G_0}$ where $K = K(D)$. In section 2, we give a geometric characterization of John disks which is the analogue of the above property of uniform domains.

We say that a domain D in $\mathbb{C}$ has the *quasidisk property* if for some fixed point $z_0 = z_0(D) \in D$ and for each $z_1 \in D$, there exists a K-quasidisk G_1 in D with $z_0, z_1 \in \overline{G_1}$, where $K = K(D)$.

THEOREM 1.2. *A bounded Jordan domain D in $\mathbb{C}$ is a c-John disk if and only if it has the quasidisk property.*

In section 3, using the above result we obtain another geometric characterization of John disks which is also the analogue of a property of uniform domains. In particular, Gehring and Martio show in [5] that a finitely connected domain D in $\mathbb{C}$ is uniform if and only if D is a QED domain, i.e., if and only if there exists a constant M such that

$$\operatorname{mod}(\Gamma) \leq M \operatorname{mod}(\Gamma_D),$$

for the families of curves Γ and Γ_D which join any pair of continua F_1 and F_2 in $\mathbb{C}$ and D, respectively. Here $\operatorname{mod}(\Gamma)$ is the *modulus* of Γ (see [2], [12]).

We say that a domain D in $\mathbb{C}$ is *M-QED with respect to $E \subset D$*, $1 \leq M < \infty$, if for each pair of disjoint continua $F_1, F_2 \subset E$

$$\operatorname{mod}(\Gamma) \leq M \operatorname{mod}(\Gamma_D), \tag{1.3}$$

where Γ and Γ_D are the families of curves joining F_1 and F_2 in $\mathbb{C}$ and in D, respectively.

THEOREM 1.4. *Suppose that D is a bounded Jordan domain in $\mathbb{C}$. Then D is M-QED with respect to all hyperbolic geodesics in D with given $z_0 \in D$ as an end point if and only if D is a c-John disk.*

2. Quasidisk Property of John disks

LEMMA 2.1. *[4, Theorem 4.1] If D is a c-John disk with a John center z_0 and if γ is a hyperbolic geodesic which joins z_1 to z_0 for $z_1 \in D$, then γ is a b-John arc for some constant b which depends only on c.*

LEMMA 2.2. *[3] and [7] Suppose that D is a Jordan domain in $\mathbb{C}$ If γ is a hyperbolic geodesic in D and if α is any curve which joins the end points of γ in D, then*

$$\ell(\gamma) \leq k\ell(\alpha),$$

where k is an absolute constant, $4.5 \leq k \leq 17.5$.

LEMMA 2.3. *Let D be a c-John disk with a John center z_0 and let γ be a hyperbolic geodesic with z_0 as one of its endpoints. If $z_1, z_2 \in \gamma$ and if z_1 separates z_0 and z_2, then*

$$\ell(\gamma(z_1, z_2)) \leq b \min(|z_1 - z_2|, d(z_1, \partial D))$$

where b is a constant which depends only on c

Proof. Fix $z_1, z_2 \in \gamma$. By Lemma 2.1,

$$\ell(\gamma(z_1, z_2)) \leq b_1\, d(z_1, \partial D) \tag{2.4}$$

for some constant b_1 which depends only on c.

If $|z_1 - z_2| \geq d(z_1, \partial D)$, then by (2.4)

$$\ell(\gamma(z_1, z_2)) \leq b_1\, |z_1 - z_2|. \tag{2.5}$$

If $|z_1 - z_2| < d(z_1, \partial D)$, then the segment $[z_1, z_2]$ joining z_1 and z_2 lies in D and

$$\ell(\gamma(z_1, z_2)) \leq c_2 \ell([z_1, z_2]) = c_2 |z_1 - z_2|, \tag{2.6}$$

by Lemma 2.2 for an absolute constant $c_2 > 0$. Hence (2.4), (2.5) and (2.6) complete the proof of Lemma 2.3 with $b = \max(b_1, c_2)$. □

Proof of Theorem 1.2. Suppose that a bounded Jordan domain D in $\mathbf{C}$ is a c-John disk with a John center z_0. Fix $z_1 \in D$ and let γ be the hyperbolic geodesic joining z_0 and z_1 in D. Fix $w_1, w_2 \in \gamma$ labeled so that w_1 separates z_0 and w_2 in γ. Then by Lemma 2.3,

$$\ell(\gamma(w_1, w_2)) \leq b|w_1 - w_2|$$

where b is a constant which depends only on c. Next if $z \in \gamma$, then z separates z_0 and z_1 in γ and by Lemma 2.3

$$\min_{j=0,1} \ell(\gamma(z_j, z)) \leq \ell(\gamma(z, z_1)) \leq bd(z, \partial D).$$

Thus γ satisfies conditions in (4.1) of [6] with $a_1 = b_1 = b$ and the construction given on [6, pp.67-68] yields a K-quasidisk G_1 with desired properties, where $K = K(a_1, b_1) = K(c)$.

Conversely, we assume that there exist a point $z_0 \in D$ and a constant K such that for each $z_1 \in D$, there is a K-quasidisk G_1 in D with $z_0, z_1 \in \overline{G_1}$. Fix $z_1 \in D$, choose a quasidisk G_1 in D corresponding to z_1 and let γ be the hyperbolic geodesic joining z_0 and z_1 in G_1. Then for all $z \in \gamma$ we have a constant $a = a(K)$ such that

$$\ell(\gamma(z, z_1)) \leq a|z - z_1| \tag{2.7}$$

and

$$\min_{j=0,1} \ell(\gamma(z_j, z)) \leq ad(z, \partial G_1) \leq ad(z, \partial D) \tag{2.8}$$

[6, Corollary 4]. Next let

$$b = \frac{\text{dia}(D)}{d(z_0, \partial D)} < \infty$$

and let $c = 2a^2 b$. We will show that

$$\ell(\gamma(z, z_1)) \leq cd(z, \partial D)$$

for all $z \in \gamma$ and hence that D is a c-John disk. We consider two cases.

Suppose first that

$$|z - z_0| \le \frac{1}{2} d(z_0, \partial D).$$

Then

$$d(z, \partial D) \ge d(z_0, \partial D) - |z - z_0| \ge \frac{1}{2} d(z_0, \partial D)$$

and hence by (2.7)

$$\begin{aligned} \ell(\gamma(z, z_1)) &\le a|z - z_1| \le a \,\mathrm{dia}(D) = ab\, d(z_0, \partial D) \\ &\le 2ab\, d(z, \partial D) \le cd(z, \partial D). \end{aligned}$$

Suppose next that

$$|z - z_0| \ge \frac{1}{2} d(z_0, \partial D).$$

If $\ell(\gamma(z_0, z)) \le \ell(\gamma(z, z_1))$, then as above

$$\begin{aligned} \ell(\gamma(z, z_1)) &\le a \mathrm{dia}(D) \le abd(z_0, \partial D) \le 2ab|z - z_0| \\ &\le 2ab\ell(\gamma(z, z_0)) \le 2a^2 bd(z, \partial D) = cd(z, \partial D). \end{aligned}$$

If $\ell(\gamma(z_0, z)) \ge \ell(\gamma(z, z_1))$, then by (2.8)

$$\ell(\gamma(z, z_1)) \le ad(z, \partial D) \le cd(z, \partial D). \quad \square$$

3. QED property of John disks

In [5], Gehring and Martio show that a simply connected proper subdomain D in $\mathbb{C}$ is a QED domain if and only if it is a quasidisk. Now we will consider the QED property for John disks in $\mathbb{C}$.

LEMMA 3.1. *[5, Remark 2.23] Suppose that D is a simply connected proper subdomain in $\mathbb{C}$. Then D is a QED domain if and only if D is a quasidisk.*

LEMMA 3.2. *[12, Theorem 10.12] Suppose that $0 < a < b$ and that E and F are disjoint sets such that every circle $S^1(t)$, $a < t < b$, meets both E and F. If G contains the annulus $A = \mathbb{B}(0,b) \setminus \overline{\mathbb{B}}(0,a)$ and if Γ is a family of curves joining E and F in G, then*

$$\operatorname{mod}(\Gamma) \geq \frac{2}{\pi} \log \frac{b}{a}.$$

LEMMA 3.3. *Suppose that D is a Jordan domain in $\overline{\mathbb{C}}$ and that γ is a hyperbolic line joining $w_1, w_2 \in \partial D$ in D. Then*

$$\frac{1}{b} \leq \frac{d(z,\alpha_1)}{d(z,\alpha_2)} \leq b, \qquad b = 3 + 2\sqrt{2}$$

for all $z \in \gamma$ where α_1 and α_2 are two components of $\partial D \setminus \{w_1, w_2\}$.

Lemma 3.3 shows that each hyperbolic line in D which joins two points on ∂D lies in the middle of D. (See [11, Exercise 1, p. 318].)

Proof of Lemma 3.3. Fix $z \in \gamma$. Then by symmetry it is sufficient to show that

$$d_1 \leq b d_2,$$

where $d_j = d(z, \alpha_j)$, $j = 1, 2$. For this we may clearly assume that $d_2 < d_1$ and hence that $d_2 = d(z, \partial D)$. Next by performing preliminary similarity mapping we may further assume that $z = 0$ and $d_1 = 1$. Choose $z_2 \in \alpha_2$ such that $|z - z_2| = d_2$, let u denote the harmonic measure of α_2 in D and set

$$f(z) = \frac{z - z_2}{1 - \overline{z}_2 z}.$$

Then $v = u \circ f^{-1}$ is positive and harmonic in $D' = f(D)$. Next fix $\omega \in \partial D'$ with $|\omega| < 1$ and let $\zeta = f^{-1}(\omega)$. Then $\zeta \in \partial D$, $|\zeta| < 1$ and thus

$$|\zeta - 0| < 1 = d(0, \alpha_1).$$

Therefore we conclude that $\zeta \in \operatorname{int}(\alpha_2)$ and hence that

$$\lim_{w \to \omega} v(w) = \lim_{z \to \zeta} u(z) = 1$$

for $w \in D'$ and $z \in D$. Finally since $0 \in \partial D'$ and $\partial D'$ contains a point which lies outside of $|w| < 1$, we see that each circle $|w| = r$ meets $\partial D'$ for $0 \leq r \leq 1$. Hence by [9, pp. 104-107],

$$v(w) \geq \frac{2}{\pi} \arcsin \frac{1 - |w|}{1 + |w|}$$

for $w \in D'$ with $|w| < 1$. In particular,

$$\frac{1}{2} = u(0) = v(-z_2) \geq \frac{2}{\pi} \arcsin \frac{1 - |z_2|}{1 + |z_2|} = \frac{2}{\pi} \arcsin \frac{1 - d_2}{1 + d_2}.$$

Thus

$$\frac{1 - d_2}{1 + d_2} \leq \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

and hence

$$d_1 = 1 \leq (3 + 2\sqrt{2}) d_2. \quad \square$$

Proof of Theorem 1.4. Suppose that a bounded Jordan domain D in $\mathbb{C}$ is a c-John disk with a John center z_0. Fix a point $z_1 \in D$ and let γ be the hyperbolic geodesic in D with end points $z_0, z_1 \in D$. Then by Theorem 1.2 there is a K-quasidisk G_1 in D such that $z_0, z_1 \in \overline{G_1}$. Thus by Lemma 3.1, G_1 is an M-QED domain where M is a constant which depends only on K, and hence only on c. Hence by [12, Theorem 6.2]

$$\operatorname{mod}(\Gamma) \leq M \operatorname{mod}(\Gamma_{G_1}) \leq M \operatorname{mod}(\Gamma_D),$$

where $\Gamma, \Gamma_{G_1}, \Gamma_D$ are the families of curves which join two disjoint subarcs F_1, F_2 of γ in $\mathbb{C}, G_1, D$, respectively. Hence D satisfies (1.3) for each pair of disjoint continua F_1, F_2 in the hyperbolic geodesics in D with given $z_0 \in D$ as an end point.

Suppose next that D is M-QED with respect to all hyperbolic geodesics in D with given $z_0 \in D$ as an end point. Fix $z_1 \in D$, $z_1 \neq z_0$ and let γ be the hyperbolic geodesic in D with end points z_0, z_1. We show first that for each $z \in \gamma$

$$\min(|z_0 - z|, |z - z_1|) \leq a d(z, \partial D) \tag{3.4}$$

for some constant $a > 1$. Suppose otherwise. Then for each constant $a > 1$, there is a point $z \in \gamma$ such that

$$\min(|z_0 - z|, |z - z_1|) > ad(z, \partial D).$$

Consider the hyperbolic line in D which contains γ with end points $w_1, w_2 \in \partial D$ and let α_1, α_2 be the two components of $\partial D \setminus \{w_1, w_2\}$. Then

$$d(z, \partial D) = \min_{j=1,2} d(z, \alpha_j)$$

for $z \in \gamma$. Thus we may assume that $d(z, \partial D) = d(z, \alpha_1)$ and by Lemma 3.3

$$d(z, \alpha_2) \leq bd(z, \partial D), \qquad b = 3 + 2\sqrt{2}. \tag{3.5}$$

Let $r = bd(z, \partial D)$ and consider the disks $\mathbb{B}(z, r), \mathbb{B}(z, \sqrt{a}r), \mathbb{B}(z, ar)$. By means of a preliminary similarity mapping we may assume that $z = 0$. Let $A = \mathbb{B}(0, ar) \setminus \overline{\mathbb{B}}(0, \sqrt{a}r)$. By hypothesis $z_0, z_1 \notin \mathbb{B}(0, ar)$. For $j = 0, 1$ let F_j denote a component of $A \cap \gamma(0, z_j)$ which joins the boundary circles of ∂A. Then by Lemma 3.2

$$\operatorname{mod}(\Gamma) \geq \operatorname{mod}(\Gamma_A) = \frac{2}{\pi} \log \sqrt{a}, \tag{3.6}$$

where Γ, Γ_A are the families of curves joining F_0 and F_1 in $\mathbb{C}$ and in A, respectively. Now let $B = \mathbb{B}(0, \sqrt{a}r) \setminus \overline{\mathbb{B}}(0, r), E = \partial\mathbb{B}(0, r)$, and $F = \partial\mathbb{B}(0, \sqrt{a}r)$. Then by (3.5), $\Gamma_B < \Gamma_D$ and hence by [12, Theorem 6.4] and [12, 7.5] we have

$$\begin{aligned} \operatorname{mod}(\Gamma_D) \leq \operatorname{mod}(\Gamma_B) &= 2\pi (\log \frac{\sqrt{a}r}{r})^{-1} \\ &= \frac{2\pi}{\log \sqrt{a}}, \end{aligned} \tag{3.7}$$

where Γ_B is the family of curves joining E and F in B and Γ_D is the family of curves joining F_0 and F_1 in D. Then, since D is M-QED with respect to γ, (3.6) and (3.7) imply that

$$\frac{2}{\pi} \log \sqrt{a} \leq \operatorname{mod}(\Gamma) \leq M \operatorname{mod}(\Gamma_D) \leq \frac{2\pi M}{\log \sqrt{a}}$$

and hence that

$$M \geq \left(\frac{\log \sqrt{a}}{\pi}\right)^2$$

which is a contradiction. Therefore for each $z \in \gamma$

$$\min(|z_1 - z|, |z_0 - z|) \leq ad(z, \partial D)$$

for some constant $a > 1$.

Next to show that D is a c-John disk we must prove that for each $z \in \gamma$,

$$|z_1 - z| < cd(z, \partial D)$$

for some constant c. For this let $d = d(z_0, \partial D)$, let $L = \max\{|z_0 - z| : z \in \partial D\}$ and let $c_1 = \max(\frac{L}{d}, a)$.

If $|z - z_1| < |z - z_0|$, then by (3.4)

$$|z - z_1| < ad(z, \partial D). \tag{3.8}$$

If $|z - z_1| > |z - z_0|$, then $|z_0 - z_1| \leq L$ and (3.4) give

$$\begin{aligned} \frac{|z - z_1|}{c_1} &\leq \frac{|z - z_0|}{c_1} + \frac{|z_0 - z_1|}{c_1} \\ &< \frac{|z - z_0|}{a} + \frac{|z_0 - z_1|}{\frac{L}{d}} \\ &< d(z, \partial D) + d. \end{aligned}$$

Now by (3.4)

$$\begin{aligned} d = d(z_0, \partial D) &\leq |z - z_0| + d(z, \partial D) \\ &\leq ad(z, \partial D) + d(z, \partial D) \\ &= (a+1)d(z, \partial D). \end{aligned}$$

Thus

$$\frac{|z - z_1|}{c_1} < d(z, \partial D) + (a+1)d(z, \partial D).$$

Hence we get

$$|z - z_1| < c_1(a+2)d(z, \partial D). \tag{3.9}$$

Therefore by (3.8) and (3.9)

$$|z - z_1| < cd(z, \partial D),$$

where $c = c_1(a+2)$. This completes the proof. □

REMARK 3.10. *In [5, Theorem 2.22], Gehring and Martio show that if D is a simply connected domain in $\mathbb{C}$, then the following conditions are equivalent:*

(1) *D is a QED domain.*

(2) *D is a uniform domain.*

Now by using an argument similar to that in the proof of Theorem 1.4 we can replace (1) by the following condition:

(1') D is M-QED with respect to all hyperbolic geodesics in D.

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