Pusan Kyŏngnam Math. J 9(1993), No. 1, pp. 143–145

ON EVALUATIONS

WON HUH

1.Introduction

We will use the following notational conventions: let (X, τ) and (Y, η) be topological spaces, and denote by $C(X;Y)$ the set of all continuous mappings of (X, τ) into (Y, η), and "a function space" will always mean a topological space equipped with the compact open topology, in which the compact open topology on $C(X;Y)$ is that having as subasis all subsets

$$(A, W) = \{f \in C(X;Y) | \vec{f}(A) \subset W\},$$

where A is τ-compact and W is η-open, and $\vec{f}(A) = \{f(a) | a \in A\}$.

The purpose of this note is to study some properties of evaluations. This note is neither intended to present substantial new results nor provide an encyclopaedic survey, but rather to give certain aspect to the subject for the choice of personal taste and preference.

2. Definition of evaluation and its basic properties

Let (X, τ) and (Y, η) be topological spaces. The mapping

$$v : C(X;Y) \times X \longrightarrow Y$$

defined by $v(f, x) = f(x)$ for each $f \in C(X;Y)$ and each $x \in X$ is called the evaluation of $C(X;Y)$.

DEFINITION. By an associate of a mapping $\alpha{:}Z \times X \longrightarrow Y$, we mean a mapping $\beta{:}Z \longrightarrow Y^X = \{f | f : X \longrightarrow Y\}$ such that for each $z \in Z$ and each $x \in X$, $(\beta(z))(x) = \alpha(z, x)$.

Received April 26, 1993

This paper was supported by the Faculty Research Fund of Pusan National University, Fy 89–93

THEOREM 1. *Let (X, τ) and (Y, η) be topological spaces. Then an evaluation $v : C(X;Y) \times X \longrightarrow Y$ is continuous if and only if for each topological space (Z, ζ), the continuity of the associate $\beta : Z \longrightarrow C(X;Y)$ of α: $Z \times X \longrightarrow Y$ implies that of α.*

Proof. : "Only if" part: Let $1_X : X \longrightarrow X$ be the identity homeomorphism; then since each mapping in the sequence $\beta \times 1_X : Z \times X \longrightarrow C(X;Y) \times X$ and $v : C(X;Y) \times X \longrightarrow Y$ is continuous, the combined mapping $v \circ (\beta \times 1_X) : Z \times X \longrightarrow Y$ is continuous. Since for each $z \in Z$ and each $x \in X$,

$$v \circ (\beta \times 1_X)(z, x) = (\beta(z))(x) = \alpha(z, x),$$

we have $v \circ (\beta \times 1_X) = \alpha$, and hence α is continuous. "If" part: Let $Z = C(X;Y)$ and let $\beta = 1_{C(X,Y)} : C(X;Y) \longrightarrow C(X;Y)$ be the identity homeomorphism; then since for each $f \in C(X;Y)$ and each $x \in X$,

$$(1_{C(X,Y)}(f))(x) = f(x) = v(f, x),$$

the condition that the continuity of the associate $1_{C(X;Y)}$ of v assures the continuity of v.

A sufficient condition for the evaluation of $C(X;Y)$ to be continuous is in the following

COROLLARY. *Let $(X; \tau)$ be a locally compact Hausdorff space, then the evaluation of $C(X;Y)$ is continuous.*

Proof. Let (Z, ζ) be any space and let $\hat{\alpha} : Z \longrightarrow C(X;Y)$, the associate of $\alpha : Z \times X \longrightarrow Y$, be continuous; we are going to show that α is continuous. To this end, let W be an η-open set containing $\alpha(z, x)$, then for each τ-compact neighbourhood A of $x \in X$,$\hat{\alpha}(z) \in (A, W)$, and since $\hat{\alpha}$ is continuous, we can find a ζ-open set H containing $z \in Z$ such that $\hat{\alpha}(H) \subset (A, W)$, so that $\alpha(H \times A) \subset W$, showing that α is continuous; By Theorem 1, the evaluation of $C(X;Y)$ is continuous.

References

1. James Dugundji, *Topology*, Allyn and Bacon,Inc.,, Boston, 1970, pp 257–267.
2. Sze-Tsen Hu, *Elements of General Topology*, Holden-Day,Inc.,, San Fransisco, 1965, pp. 150–166.

Department of Mathematics
Pusan National University
Pusan 609–735, Korea