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ON ISOMETRIC DILATION AND COMMON NONCYCLIC VECTORS FOR FAMILIES OF OPERATORS*

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1. Introduction Suppose $\mathcal{H}$ is a separable, infinite dimensional, complex Hilbert spaces and $\mathcal{L}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the smallest subalgebra of $\mathcal{L}(\mathcal{H})$ that cointains T and $1_{\mathcal{H}}$ and is closed in the ultraweak operator topology ($\mathcal{A}_T$ is called a dual algebra generated by T). Moreover, let $Q_{\mathcal{A}_T}$ denote the quotient space $\mathcal{C}_1(\mathcal{H})/{}_\perp\mathcal{A}_T$, where $\mathcal{C}_1(\mathcal{H})$ is the trace class ideal in $\mathcal{L}(\mathcal{H})$ under the trace norm, and ${}_\perp\mathcal{A}_T$ denotes the pre-annihilator of $\mathcal{A}_T$ in $\mathcal{C}_1(\mathcal{H})$. For a brief notation, we shall denote $Q_{\mathcal{A}_T}$ by Q_T. One knows (cf.[4]) that $\mathcal{A}_T$ is the dual space of Q_T and that the duality is given by

$$(1) \qquad \langle A,[L]\rangle = tr(AL), \quad A \in \mathcal{A}_T, [L] \in Q_T.$$

The Banach space Q_T is called a predual of $\mathcal{A}_T$. If x and y are vectors in $\mathcal{H}$, we can write $x \otimes y$ for the rank one operator in $\mathcal{C}_1(\mathcal{H})$ defined by

$$(2) \qquad (x \otimes y)(u) = (u,y)x \quad for \quad all \quad u \in \mathcal{H},$$

then $[x \otimes y] \in Q_T$ and an easy calculation shows that for any A in $\mathcal{A}_T$ we have

$$(3) \qquad \langle A,[x \otimes y]\rangle = tr(A(x \otimes y)) = (Ax,y).$$

The theory of dual algebras is applied to the study of invariant subspaces, dilation theory, and reflexivity. The classes $\mathcal{A}_{m,n}$ (to be defined

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in section 2) were defined by H.Bercovici, C.Foias and C.Pearcy in [2]. Also these classes are closely related to the study of the theory of dual algebras. S.Brown, B.Chevreau, G.Exner and C.Pearcy [6], [7], [9] obtained topological criteria and geometric criteria for membership in the class $A_{\aleph_0}$ or $A_{1,\aleph_0}$. In [8] B.Chevreau and C.Pearcy studied common noncyclic vectors for families of operators (to be defined in section 2). In a sequel to this study, Han-soo Kim and Hae-gyu Kim [11] obtained an equivalent condition for membership in the classes $\mathcal{A}^{l}_{m,n}$ (to be defined in section 2) by using minimal coisometric extension of a contraction operator in $\mathcal{L}(\mathcal{H})$.
In this paper, we consider an isometric dilation V of a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{H}$ is a semi-invariant subspace (to be defined in section 2) for V and, using techniques similar to the ones in [7], we establish an equivalent condition for membership in the classes $\mathcal{A}^{l}_{m,n}$.

2. Fundamental theories and terminologies In this section, we introduce some fundamental theorems and notations on the theory of dual algebras, which we shall use in this work. The notation and terminology employed herein agree with those in [3], [5], [14]. We shall denote by D the open unit disc in the complex plane C, and we write T for the boundary of D. The space $L^p = L^p(T), 1 \le p \le \infty$, is the usual Lebesgue function space relative to normalized Lebesgue measure m on T. The space $H^p = H^p(T), 1 \le p \le \infty$, is the usual Hardy space. It is well-known that the space H^∞ is the dual space of L^1/H^1_0, where

$$(4) \qquad H^1_0 = \{f \in L^1 : \int_0^{2\pi} f(e^{it})e^{int}dt = 0, \quad for \quad n = 0,1,2,\cdots\}$$

and the duality is given by the pairing

$$(5) \qquad \langle f,[g]\rangle = \int_T fg dm \quad for \quad f \in H^\infty, [g] \in L^1/H^1_0.$$

Recall that any contraction T can be written as a direct sum $T = T_1 \oplus T_2$, where T_1 is a completely nonunitary contraction and T_2 is a unitary operator. If T_2 is absolutely continuous or acts on the space (0), T will be called an *absolutely continuous contraction.* The following Foias-Sz.Nagy functional calculus [3, Theorem 4.1] provides a good relationship between the function space H^∞ and a dual algebra $\mathcal{A}_T$.

Theorem 2.1.([3. Theorem 4.1])*Let T be an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$. Then there is an algebra homomorphism Φ_T : $H^\infty \to \mathcal{A}_T$ defined by $\Phi_T(f) = f(T)$ such that*
(a) $\Phi_T(1) = 1_{\mathcal{H}}, \quad \Phi_T(\xi) = T$,
(b) $\|\Phi_T(f)\| \le \|f\|_\infty, \quad f \in H^\infty$,
(c) *Φ_T is continuous if both H^∞ and $\mathcal{A}_T$ are given their weak* topologies,*
(d) *the range of Φ_T is weak* dense in $\mathcal{A}_T$,*
(e) *there exists a bounded, linear, one-to-one map $\phi_T : Q_T \to L^1/H_0^1$ such that ${\phi_T}^* = \Phi_T$, and*
(f) *if Φ_T is an isometry, then Φ_T is a weak* homeomorphism of H^∞ onto $\mathcal{A}_T$ and ϕ_T is an isometry of Q_T onto L^1/H_0^1.*

Definition 2.2.([12]) *Let $\mathcal{A} \subset \mathcal{L}(\mathcal{H})$ be a dual algebra and let m and n be any cardinal numbers such that $1 \le m, n \le \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(A_{m,n})$ if every $m \times n$ system of simultaneous equations of the form*

$$[x_i \otimes y_j] = [L_{ij}], \quad 0 \le i < m, 0 \le j < n, \tag{6}$$

where $\{[L_{ij}]\}_{\substack{0\le i<m \\ 0\le j<n}}$ is an arbitrary $m\times n$ array from $Q_{\mathcal{A}}$, has a solution consisting of a pair of sequences $\{x_i\}_{0\le i<m}, \{y_j\}_{0\le j<n}$ of vectors from $\mathcal{H}$. For brief notation, we shall denote $(A_{n,n})$ by (A_n). We denote by $A = A(\mathcal{H})$ the class of all absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foias-Sz Nagy functional calculus Φ_T . $\mathcal{H}^\infty \to \mathcal{A}_T$ is an isometry. Furthermore, if m and n are cardinal numbers such that $1 \le m, n \le \aleph_0$, we denote by $A_{m,n} = A_{m,n}(\mathcal{H})$ the set of all T in $A(\mathcal{H})$ such that the singly generated dual algebra $\mathcal{A}_T$ has property $(A_{m,n})$.

Definition 2.3.([8]) *If $\{T_\alpha\}_{\alpha\in I}$ is a family of operators in $\mathcal{L}(\mathcal{H})$ and x is a nonzero vector in $\mathcal{H}$ such that for each $\alpha \in I$,*

$$\mathcal{M}_\alpha = \bigvee_{n=0}^{\infty} T_\alpha^n x \ne \mathcal{H}, \tag{7}$$

then x is said to be a common noncyclic vector for the family $\{T_\alpha\}_{\alpha\in I}$.

Definition 2.4.([3]) *$\mathcal{K}$ will be denoted by an arbitrary complex Hilbert space such that dim $\mathcal{K} \leq \aleph_0$. If $T \in \mathcal{L}(\mathcal{K})$ and $\mathcal{M} \subset \mathcal{K}$ is a* ***semi-invariant*** *subspace for T (i.e., there exist $\mathcal{N}_1$ and $\mathcal{N}_2$ in Lat(T) such that $\mathcal{M} = \mathcal{N}_1 \ominus \mathcal{N}_2$ and $\mathcal{N}_2 \subset \mathcal{N}_1$), we write $T_{\mathcal{M}}$ for the compression of T to $\mathcal{M}$. In other words,*

$$T_{\mathcal{M}} = P_{\mathcal{M}} T|_{\mathcal{M}}, \tag{8}$$

where $P_{\mathcal{M}}$ is the orthogonal projection whose range is $\mathcal{M}$.

We shall employ the notation $C_{\cdot 0} = C_{\cdot 0}(\mathcal{H})$ for the class of all (completely nonunitary) contractions T in $\mathcal{L}(\mathcal{H})$ such that the sequences $\{T^{*n}\}$ converges to zero in the strong operator topology and is denoted by,as usual,$C_{0\cdot} = (C_{\cdot 0})^*$.

To establish our results, it will be convenient to use the minimal isometric dilation theorem [14]: every contraction T in $\mathcal{L}(\mathcal{H})$ has a minimal isometric dilation V that is unique up to isomorphism.

Given such T and V, one knows that there exists a canonical decomposition of the isometry V as

$$V = S \oplus R \tag{9}$$

corresponding to a decomposition of the space

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R}, \tag{10}$$

where, if $\mathcal{S} \neq (0)$, S is a unilateral shift operator of some multiplicity in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \neq (0)$, R is a unitary operator in $\mathcal{L}(\mathcal{R})$. Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be (0). ([7])

Notational convention 2.5. *Given such T and V, the projection of $\mathcal{K}$ onto $\mathcal{S}$ will be denoted by Q and the projection of $\mathcal{K}$ onto $\mathcal{R}$ will be denoted by A, so $Q = 1_{\mathcal{K}} - A$ and every vector x in $\mathcal{K}$ may be written uniquely as*

$$x = Qx + Ax = Qx \oplus Ax. \tag{11}$$

Moreover, the projection of $\mathcal{K}$ onto the subspace $\mathcal{H}$ will be denoted by P.

Convention 2.6. *In this paper, $V \in \mathcal{L}(\mathcal{K})$ always denotes an isometric dilation of a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $\mathcal{H}$ is a semi-invariant subspace for V and that $V\mathcal{H} \subset \mathcal{H}$.*

Since the polynomials are sequentially weak dense in $H^\infty(T)$, we have the following lemma.*

Lemma 2.7.([15], Lemma 4.5) *If T is absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and V is absolutely continuous, then for every vector x in $\mathcal{H}$ and for every function h in $H^\infty(T)$,*

$$\begin{aligned} h(T)x = h(V)x &= h(S)(QX) \oplus h(R)(Ax) \\ &= Q(h(T)x) \oplus A(h(T)x). \end{aligned}$$

Hence we have

$$h(S)(Qx) = Q(h(T)x), \quad h(R)(Ax) = A(h(T)x).$$

Lemma 2.8.([15, Lemma 4.6]) *Suppose $T \in A(\mathcal{H})$ and V is absolutely continuous. Then $V \in A(\mathcal{K}), \Phi_T \circ \Phi_V^{-1}$ is an isometry and* weak* homeomorphism from $\mathcal{A}_V$ onto $\mathcal{A}_T$,and $j = \varphi_B^{-1} \circ \varphi_T$ *is a linear isometry of Q_T onto Q_B. Moreover,*

$$j([C_\lambda]_T) = [C_\lambda]_V, \quad \lambda \in D \tag{12}$$

and

$$j([x \otimes y]_T) = [x \otimes y]_V, \quad x, y \in \mathcal{H}. \tag{13}$$

Lemma 2.9.([15, Lemma 4.7]) *If T belongs to $A(\mathcal{H})$ and $V \in \mathcal{L}(\mathcal{K})$ is absolutely continuous, $x, y \in \mathcal{H}$, and $w, z \in \mathcal{K}$, then*

$$\|[x \otimes y]_T\| = \|[x \otimes y]_V\|, \tag{14}$$

$$[x \otimes z]_V = [x \otimes Pz]_V, \tag{15}$$

and

$$[w \otimes z]_V = [Qw \otimes Qz]_V + [Aw \otimes Az]_V. \tag{16}$$

3. An equivalent condition for membership in $\mathcal{A}^l_{m,n}(\mathcal{H})$

Definition 3.1. *([10, Definition 3.1.1]) Let m, n and l be any cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$. We denote by $A^l_{m,n}(\mathcal{H})$ the class of all set $\{T_k\}^l_{k=1}$ such that T_k belongs to $A(\mathcal{H})$ for all $k = 1, 2, \cdots, l$, and that every $m \times n \times l$ system of simultaneous equations of the form*

$$[x_i \otimes y^{(k)}_j]_{T_k} = [L^{(k)}_{ij}]_{T_k}, \tag{17}$$

where $\{[L^{(k)}_{ij}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from Q_{T_k} for each $1 \leq k \leq l$, has a solution consisting of a pair of sequences $\{x_i\}_{0 \leq i < m}, \{y^{(k)}_j\}_{\substack{0 \leq j < n \\ 1 \leq k \leq l}}$ of vectors from $\mathcal{H}$.

Example 3.2. *If $\{T_k\}^\infty_{k=1}$ are in the class $A_{\aleph_0} \cap C_{\cdot 0}$, then $\{T_k\}^\infty_{k=1} \in A^{\aleph_0}_{1,1}$,
by [11, Remark 3.2].*

We are now ready to prove our main theorem.

Theorem 3.3. *Suppose m, n and l are cardinal numbers such that $1 \leq m, n, l \leq \aleph_0$ and $T_k \in A(\mathcal{H})$ and $V_k \in \mathcal{L}(\mathcal{S} \oplus \mathcal{R})$ is absolutely continuous for k, $1 \leq k \leq l$. Then the followings are equivalent:*
(1) The set $\{T_k\}^l_{k=1} \in A^l_{m,n}$.
(2) For $\{[L^{(k)}_{ij}]_{T_k}\}_{\substack{0 \leq i < m \\ 0 \leq j < n}} \subset Q_{T_k}$ for k, $1 \leq k \leq l$, there exists a Cauchy sequence $\{x_{i,p}\}^\infty_{p=1}$ in $\mathcal{H}$ and sequences $\{w^{(k)}_{j,p}\}^\infty_{p=1}$ in $\mathcal{S}$ and $\{b^{(k)}_{j,p}\}^\infty_{p=1}$ in $\mathcal{R}$ such that $\{w^{(k)}_{j,p} + b^{(k)}_{j,p}\}$ is bounded and

$$\|(\varphi^{-1}_{V_k} \circ \varphi_{T_k})([L^{(k)}_{ij}]_{T_k}) - [x_{i,p} \otimes (w^{(k)}_{j,p} + b^{(k)}_{j,p})]_{V_k}\| \xrightarrow{p} 0.$$

Proof. $(1) \Rightarrow (2)$: Trivial.
$(2) \Rightarrow (1)$: *Let us* $v_{j,p}^{(k)} = P(w_{j,p}^{(k)} + b_{j,p}^{(k)}), p \in N$, *where* P *is an orthogonal projection from* $\mathcal{K}$ *onto* $\mathcal{H}$. *Since* $\{v_{j,p}^{(k)}\}_{p=1}^{\infty}$ *is bounded, we may suppose w.l.o.g, that* $\{v_{j,p}^{(k)}\}_{p=1}^{\infty}$ *converges weakly to* v_j^k. *Moreover, since* $\{x_{i,p}\}_{p=1}^{\infty}$ *is a Cauchy sequence, we have* $\{x_{i,p}\}$ *converges strongly to* x_i.

$$\begin{aligned} &\|[x_i \otimes v_{j,p}^{(k)}]_{T_k} - [x_{i,p} \otimes v_{j,p}^{(k)}]_{T_k}\| \\ &= \|[(x_i - x_{i,p}) \otimes v_{j,p}^{(k)}]_{T_k}\| \\ &\le \|x_i - x_{i,p}\| \cdot \|v_{j,p}^{(k)}\| \xrightarrow{p} 0. \end{aligned}$$

Also from (13) *and* (15), *with* $j_k = \varphi_{V_k}^{-1} \circ \varphi_{T_k}$, we have

$$\begin{aligned} &\|[L_{ij}^{(k)}]_{T_k} - [x_{i,p} \otimes v_{j,p}^{(k)}]_{T_k}\| \\ &= \|\varphi_{V_k}^{-1} \circ \varphi_{T_k}([L_{ij}^{(k)}]_{T_k}) - [x_{i,p} \otimes v_{j,p}^{(k)}]_{V_k}\| \\ &= \|\varphi_{V_k}^{-1} \circ \varphi_{T_k}([L_{ij}^{(k)}]_{T_k}) - [x_{i,p} \otimes (w_{j,p}^{(k)} + b_{j,p}^{(k)})]_{V_k}\| \xrightarrow{p} 0. \end{aligned}$$

Then

$$\begin{aligned} &\|[L_{ij}^{(k)}]_{T_k} - [x_i \otimes v_{j,p}^{(k)}]_{T_k}\| \\ &\le \|[L_{ij}^{(k)}]_{T_k} - [x_{i,p} \otimes v_{j,p}^{(k)}]_{T_k}\| \\ &\quad + \|[x_{i,p} \otimes v_{j,p}^{(k)}]_{T_k} - [x_i \otimes v_{j,p}^{(k)}]_{T_k}\| \xrightarrow{p} 0. \end{aligned}$$

Hence we have $[L_{ij}^{(k)}]_{T_k} = [x_i \otimes v_j^{(k)}]_{T_k}, \ 0 \le i < m, 0 \le j < n, 1 \le k \le l$.
Therefore the proof is complete.

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