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FUZZY CHARACTERISTIC SUBALGEBRAS/IDEALS OF A BCK-ALGEBRA

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In this paper we introduce the concept of fuzzy characteristic subalgebras/ideals of a BCK-algebra. A fuzzy characteristic subalgebra/ideal is characterized in terms of its level subalgebras/ideals.

Recall that a BCK-algebra is a non-empty set X with a binary operation $*$ and a constant 0 satisfying the axioms

BCK-1 $((x * y) * (x * z)) * (z * y) = 0$,

BCK-2 $(x * (x * y)) * y = 0$,

BCK-3 $x * x = 0$,

BCK-4 $0 * x = 0$,

BCK-5 $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z \in X$. We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x * y = 0$. A mapping $f : X \to Y$ of BCK-algebras is called a homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. A BCK-algebra X is said to be bounded if there exists an element $1 \in X$ such that $x \leq 1$ for all $x \in X$. Throughout X will denote a BCK-algebra and $Aut(X)$ the set of all automorphisms of X. A nonempty subset S of X is called a subalgebra of X if $x * y \in S$ for any $x, y \in S$. An ideal of X is a subset I containing 0 such that if $x * y$ and y are in I then so is x.

We now review some fuzzy logic concepts. We refer the reader to [3] and [5] for complete details. A fuzzy subset of X is a function $\mu : X \to [0, 1]$. It is a fuzzy subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$, and it is a fuzzy ideal of X if $\mu(0) \geq \mu(x)$ and $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$. Given a fuzzy subset μ and $t \in [0, 1]$, let $\mu_t = \{x \in X | \mu(x) \geq t\}$. This could be an empty set. It was shown in [5] that a fuzzy subset μ of X is a fuzzy subalgebra/ideal of X if and only if for each $t \in [0, 1], \mu_t$ is either empty or a subalgebra/ideal of X. The subalgebras/ideals $\mu_t, t \in [0, 1]$, are called level subalgebras/ideals of μ.

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DEFINITION 1. If μ is a fuzzy subalgebra/ideal of X and θ is a map from X into itself, we define a map $\mu^\theta : X \to [0,1]$ by $\mu^\theta(x) = \mu(\theta(x))$ for every $x \in X$.

If μ is a fuzzy subalgebra of X and θ is an endomorphism of X, then

$$\begin{aligned}\mu^\theta(x*y) &= \mu(\theta(x*y)) = \mu(\theta(x)*\theta(y))\\ &\geq \min\{\mu(\theta(x)), \mu(\theta(y))\}\\ &= \min\{\mu^\theta(x), \mu^\theta(y)\}\end{aligned}$$

for all $x, y \in X$. Hence we have the following theorem.

THEOREM 1. *If μ is a fuzzy subalgebra of X and θ is an endomorphism of X, then μ^θ is a fuzzy subalgebra of X.*

PROPOSITION 1. *If μ is a fuzzy subalgebra of X and θ is an endomorphism of X, then $\mu^\theta(0) \geq \mu^\theta(x)$ for all $x \in X$.*

Proof. We have that $\mu^\theta(x) = \mu(\theta(x)) = \min\{\mu(\theta(x)), \mu(\theta(x))\} \leq \mu(\theta(x)*\theta(x)) = \mu(\theta(x*x)) = \mu^\theta(0)$ for any $x \in X$.

PROPOSITION 2. *Let μ be a fuzzy ideal of X and $\theta : X \to X$ an onto homomorphism. Then the following hold for all $x, y, z \in X$,*

(1) *if $x \leq y$ then $\mu^\theta(x) \geq \mu^\theta(y)$.*
(2) *$\mu^\theta(x*y) \geq \min\{\mu^\theta(x*z), \mu^\theta(z*y)\}$.*
(3) *if $\mu^\theta(x*y) = \mu^\theta(0)$ then $\mu^\theta(x) \geq \mu^\theta(y)$.*
(4) *$\min\{\mu^\theta(x*y), \mu^\theta(y)\} = \min\{\mu^\theta(x), \mu^\theta(y)\}$.*
(5) *if X is bounded then $\min\{\mu^\theta(x), \mu^\theta(1*x)\} = \mu^\theta(1)$.*
(6) *if $x \leq y$ then $\mu^\theta(y) = \min\{\mu^\theta(y*x), \mu^\theta(x)\}$.*

Proof. (1) If $x \leq y$ then $x*y = 0$. Hence

$$\begin{aligned}\mu^\theta(y) &= \mu(\theta(y))\\ &= \min\{\mu(0), \mu(\theta(y))\}\\ &= \min\{\mu(\theta(0)), \mu(\theta(y))\}\\ &= \min\{\mu(\theta(x*y)), \mu(\theta(y))\}\\ &= \min\{\mu(\theta(x)*\theta(y)), \mu(\theta(y))\}\\ &\leq \mu(\theta(x) = \mu^\theta(x).\end{aligned}$$

(2) From BCK-1 and (1), it follows that $\mu^\theta((x*y)*(x*z)) \geq \mu^\theta(z*y)$. Hence

$$\begin{aligned}
\mu^\theta(x*y) &= \mu(\theta(x*y)) \\
&= \mu(\theta(x)*\theta(y)) \\
&\geq \min\{\mu((\theta(x)*\theta(y))*(\theta(x)*\theta(z))), \mu(\theta(x)*\theta(z))\} \\
&= \min\{\mu^\theta((x*y)*(x*z)), \mu^\theta(x*z)\} \\
&\geq \min\{\mu^\theta(z*y), \mu^\theta(x*z)\}.
\end{aligned}$$

(3) Assume that $\mu^\theta(x*y) = \mu^\theta(0)$. Then

$$\begin{aligned}
\mu^\theta(x) &= \mu(\theta(x)) \\
&\geq \min\{\mu(\theta(x)*\theta(y)), \mu(\theta(y))\} \\
&= \min\{\mu(\theta(x*y)), \mu(\theta(y))\} \\
&= \min\{\mu^\theta(x*y), \mu^\theta(y)\} \\
&= \min\{\mu^\theta(0), \mu^\theta(y)\} \\
&= \min\{\mu(\theta(0)), \mu(\theta(y))\} \\
&= \min\{\mu(0), \mu(\theta(y))\} \\
&= \mu(\theta(y)) = \mu^\theta(y).
\end{aligned}$$

(4) Since $x*y \leq x$, we have $\mu^\theta(x*y) \geq \mu^\theta(x)$ by (1). Hence

$$\begin{aligned}
\mu^\theta(x) &= \mu(\theta(x)) \\
&\geq \min\{\mu(\theta(x)*\theta(y)), \mu(\theta(y))\} \\
&= \min\{\mu(\theta(x*y)), \mu(\theta(y))\} \\
&= \min\{\mu^\theta(x*y), \mu^\theta(y)\} \\
&\geq \min\{\mu^\theta(x), \mu^\theta(y)\}
\end{aligned}$$

and so $\min\{\mu^\theta(x), \mu^\theta(y)\} = \min\{\mu^\theta(x*y), \mu^\theta(y)\}$.

(5) If X is bounded then by (1),

$$\mu^\theta(1) \leq \min\{\mu^\theta(x), \mu^\theta(1*x)\}.$$

On the other hand,

$$\begin{aligned}\mu^\theta(1) &= \mu(\theta(1)) \\ &\geq \min\{\mu(\theta(1) * \theta(x)), \mu(\theta(x))\} \\ &= \min\{\mu(\theta(1 * x)), \mu(\theta(x))\} \\ &= \min\{\mu^\theta(1 * x), \mu^\theta(x)\}.\end{aligned}$$

Thus (5) is true.

(6) is obtained from (1) and (4).

THEOREM 2. *If μ is a fuzzy ideal of X and $\theta : X \to X$ is an onto homomorphism, then μ^θ is a fuzzy ideal of X.*

Proof. We have that $\mu^\theta(x) = \mu(\theta(x)) \leq \mu(0) = \mu(\theta(0)) = \mu^\theta(0)$ for all $x \in X$. Next for any $x, y \in X$,

$$\mu^\theta(x) = \mu(\theta(x)) \geq \min\{\mu(\theta(x) * y), \mu(y)\},$$

because μ is a fuzzy ideal. Since θ is onto, there exists $z \in X$ such that $\theta(z) = y$. Hence

$$\begin{aligned}\mu^\theta(x) &\geq \min\{\mu(\theta(x) * y), \mu(y)\} \\ &= \min\{\mu(\theta(x) * \theta(z)), \mu(\theta(z))\} \\ &= \min\{\mu(\theta(x * z)), \mu(\theta(z))\} \\ &= \min\{\mu^\theta(x * z), \mu^\theta(z)\}.\end{aligned}$$

As y is an arbitrary element of X, the above result is true for any $z \in X$, i. e., $\mu^\theta(x) \geq \min\{\mu^\theta(x * z), \mu^\theta(z)\}$ for all $x, z \in X$. Hence μ^θ is a fuzzy ideal of X.

DEFINITION 2. A subalgebra/ideal K of X is called a characteristic subalgebra/ideal if $\theta(K) = K$ for all $\theta \in Aut(X)$.

DEFINITION 3. A fuzzy subalgebra/ideal μ of X is called a fuzzy characteristic subalgebra/ideal of X if $\mu(\theta(x)) = \mu(x)$ for all $x \in X$ and all $\theta \in Aut(X)$.

THEOREM 3. *Let μ be a fuzzy characteristic subalgebra of X. Then each level subalgebra of μ is a characteristic subalgebra of X.*

Proof. Let $t \in Im(\mu), \theta \in Aut(X)$ and $x \in \mu_t$. Since μ is a fuzzy characteristic subalgebra of X, we have $\mu(\theta(x)) = \mu(x) \geq t$. It follows that $\theta(x) \in \mu_t$ and hence $\theta(\mu_t) \subseteq \mu_t$. To prove the reverse inclusion, let $x \in \mu_t$ and let $y \in X$ be such that $\theta(y) = x$. Then $\mu(y) = \mu(\theta(y)) = \mu(x) \geq t$, whence $y \in \mu_t$. It follows that $x = \theta(y) \in \theta(\mu_t)$, so that $\mu_t \subseteq \theta(\mu_t)$. Thus $\mu_t, t \in Im(\mu)$, is a characteristic subalgebra of X.

The following lemma is obvious, and we omit the proof.

LEMMA 1. *Let μ be a fuzzy subalgebra/ideal of X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

Now we prove the converse of Theorem3.

THEOREM 4. *Let μ be a fuzzy subalgebra of X. If each level subalgebra of μ is a characteristic subalgebra of X, then μ is a fuzzy characteristic subalgebra of X.*

Proof. Let $x \in X, \theta \in Aut(X)$ and $\mu(x) = t$. Then $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$, by Lemma 1. Since $\theta(\mu_t) = \mu_t$ by hypothesis, we have $\theta(x) \in \mu_t$ and hence $\mu(\theta(x)) \geq t$. Let $s = \mu(\theta(x))$. If possible, let $s > t$. Then $\theta(x) \in \mu_s = \theta(\mu_s)$. Since θ is one-one, it follows that $x \in \mu_s$, which is a contradiction. Hence $\mu(\theta(x)) = t = \mu(x)$, showing that μ is a fuzzy characteristic subalgebra of X.

The proofs of the following theorems are similar to that of Theorems 3 and 4.

THEOREM 5. *If μ is a fuzzy characteristic ideal of X then each level ideal of μ is a characteristic ideal of X*

THEOREM 6. *Let μ be a fuzzy ideal of X. If each level ideal of μ is a characteristic ideal of X then μ is a fuzzy characteristic ideal of X.*

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