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FUZZY PRIME IDEALS IN Γ-RINGS

YOUNG BAE JUN AND CHONG YUN LEE

In [14], Zadeh introduced the notion of a fuzzy subset μ of a set S as a function from S into [0, 1]. Rosenfeld [12] applied this concept to the theory of groupoids and groups. Kuroki [7, 8] has studied fuzzy ideals, fuzzy bi-ideals and fuzzy semiprime ideals in semigroups. Liu [9], Mukherjee and Sen [10] and Swamy and Swamy [13] have studied fuzzy ideals and fuzzy prime ideals of a ring.

This paper is a continuation of [5] and [6]. It was shown in [6] that a Γ-homomorphic image of a fuzzy ideal which has the sup property is a fuzzy ideal. But it holds without assuming the sup property. We first prove that a Γ-homomorphic image of a fuzzy ideal is also a fuzzy ideal without assuming the sup property. In [5], the first author investigated the fuzzy prime ideals in Γ-rings. Secondly, we study more properties on fuzzy prime ideals of Γ-rings.

DEFINITION 1 ([1]). If $M = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ are additive abelian groups, and for all x, y, z in M and all α, β in Γ, the following conditions are satisfied

(1) $x\alpha y$ is an element of M,
(2) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta)y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$,
(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$,

then M is called a Γ-ring.

Through this paper M and M' denote Γ-rings, and 0_M and $0_{M'}$ denote the zero elements of M and M' respectively.

DEFINITION 2 ([1]) A subset A of M is a left (right) ideal of M if A is an additive subgroup of M and

$$M\Gamma A = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in A\}(A\Gamma M)$$

is contained in A. If A is both a left and a right ideal, then A is a two-sided ideal, or simply an ideal of M.

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DEFINITION 3 ([2]). An ideal P of M is said to be prime if for every ideals A, B of M, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

PROPOSITION 1 ([2]). *Let P be an ideal of M. Then the following are equivalent:*

(a) *P is a prime ideal of M.*

(b) *For all $x, y \in M$, $x\Gamma M\Gamma y \subseteq P$ implies $x \in P$ or $y \in P$.*

PROPOSITION 2 ([3]). *Let I be an ideal of M. If P is a prime ideal of M, then $P \cap I$ is a prime ideal of I.*

DEFINITION 4 ([1]). A mapping $\theta : M \to M'$ is called a Γ-homomorphism if $\theta(x+y) = \theta(x) + \theta(y)$ and $\theta(x\alpha y) = \theta(x)\alpha\theta(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

DEFINITION 5 ([12]). Let $\theta : M \to M'$ be any function and let μ be any fuzzy set in M. The fuzzy set η in M' defined by

$$\eta(y) = \begin{cases} \sup\limits_{x \in \theta^{-1}(y)} \mu(x) & \text{if } \theta^{-1}(y) \neq \emptyset, y \in M', \\ 0 & \text{otherwise,} \end{cases}$$

is called the image of μ under θ, denoted by $\theta(\mu)$.

DEFINITION 6 ([6]). A fuzzy set μ in M is called a fuzzy left (right) ideal of M if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(5) $\mu(x\alpha y) \geq \mu(y)$ $(\mu(x\alpha y) \geq \mu(x))$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

A fuzzy set μ in M is called a fuzzy ideal of M if μ is both a fuzzy left and a fuzzy right ideal of M.

We note that μ is a fuzzy ideal of M if and only if

(4) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,

(6) $\mu(x\alpha y) \geq \max\{\mu(x), \mu(y)\}$,

for all $x, y \in M$ and all $\alpha \in \Gamma$.

THEOREM 1. *Let $\theta : M \to M'$ be an onto Γ-homomorphism. If μ is a fuzzy ideal of M, then $\theta(\mu)$ is a fuzzy ideal of M'.*

Proof. Let $x', y' \in M'$ and $\alpha \in \Gamma$. Then there exist $x, y \in M$ such that $\theta(x) = x'$ and $\theta(y) = y'$. Then

$$\begin{aligned}
\theta(\mu)(x'-y') &= \sup_{\theta(z)=x'-y'} \mu(z) \\
&\geq \sup_{\substack{\theta(x)=x' \\ \theta(y)=y'}} \mu(x-y) \\
&\geq \sup_{\theta(x)=x'} \sup_{\theta(y)=y'} \min\{\mu(x), \mu(y)\} \\
&= \min\{\sup_{\theta(x)=x'} \mu(x), \sup_{\theta(y)=y'} \mu(y)\} \\
&= \min\{\theta(\mu)(x'), \theta(\mu)(y')\},
\end{aligned}$$

and

$$\begin{aligned}
\theta(\mu)(x'\alpha y') &= \sup_{\theta(z)=x'\alpha y'} \mu(z) \\
&\geq \sup_{\substack{\theta(x)=x' \\ \theta(y)=y'}} \mu(x\alpha y) \\
&\geq \sup_{\substack{\theta(x)=x' \\ \theta(y)=y'}} \max\{\mu(x), \mu(y)\} \\
&= \max\{\sup_{\theta(x)=x'} \mu(x), \sup_{\theta(y)=y'} \mu(y)\} \\
&= \max\{\theta(\mu)(x'), \theta(\mu)(y')\}.
\end{aligned}$$

This completes the proof.

DEFINITION 7 ([5]). Let μ and ν be fuzzy sets in M and let $\alpha \in \Gamma$. The product $\mu\Gamma\nu$ is defined by $\mu\Gamma\nu(x) = \sup_{x=y\alpha z} \min\{\mu(y), \nu(z)\}$ and $\mu\Gamma\nu(x) = 0$ if x is not expressible as $x = y\alpha z$.

DEFINITION 8 ([5]). A fuzzy ideal μ of M is said to be prime if

(7) μ is not a constant function,

(8) for any fuzzy ideals ν, ρ in M, $\nu\Gamma\rho \subseteq \mu$ implies $\nu \subseteq \mu$ or $\rho \subseteq \mu$.

LEMMA 1 ([5]). *If μ is any nonconstant fuzzy set in M, then μ is a fuzzy prime ideal of M if and only if $Im(\mu) = \{1, t\}$ where $t \in [0, 1)$ and the ideal $M_\mu = \{x \in M | \mu(x) = 1\}$ is prime.*

THEOREM 2. *Let μ be a fuzzy ideal of M such that $1 \in Im(\mu)$ and let ν be a fuzzy prime ideal of M. Then $\mu \cap \nu$ is a fuzzy prime ideal of the Γ-ring $M_\mu = \{x \in M | \mu(x) = 1\}$.*

Proof. Since ν is a fuzzy prime ideal of M, it follows from Lemma 1 that there exists $t \in [0, 1)$ such that

$$\nu(x) = \begin{cases} 1 & \text{if } x \in M_\nu, \\ t & \text{otherwise,} \end{cases}$$

where $M_\nu = \{x \in M | \nu(x) = 1\}$. As M_ν is a prime ideal of M, $M_\mu \cap M_\nu$ is a prime ideal of M_μ. Now

$$(\mu \cap \nu)(x) = \begin{cases} 1 & \text{if } x \in M_\mu \cap M_\nu, \\ t & \text{if } x \in M_\mu - (M_\mu \cap M_\nu). \end{cases}$$

Consequently $\mu \cap \nu$ is a fuzzy prime ideal of M_μ.

THEOREM 3. *Let μ be any fuzzy ideal of M. If $Im(\mu) = \{t_0, t_1, ..., t_m\}$ where $t_0 > t_1 > ... > t_m$ and each μ_{t_i} is a prime ideal of M, then $\mu(x\alpha y\beta z) = \max\{\mu(x), \mu(y), \mu(z)\}$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.*

Proof. Let $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Without loss of generality, we may assume that $\max\{\mu(x), \mu(y), \mu(z)\} = \mu(z) = t_i, 0 \le i \le m$. Since μ is a fuzzy ideal of M, it follows that

$$\begin{aligned} \mu(x\alpha y\beta z) &\ge \max\{\mu(x\alpha y), \mu(z)\} \\ &\ge \max\{\mu(x), \mu(y), \mu(z)\} \\ &= t_i. \end{aligned}$$

Suppose that $\mu(x\alpha y\beta z) > t_i$. Then $\mu(x\alpha y\beta z) \in \{t_0, t_1, ..., t_{i-1}\}$, and hence $x\alpha y\beta z \in \mu_{t_{i-1}}$. As $\mu_{t_{i-1}}$ is a prime ideal, it follows from Proposition 1 that $x \in \mu_{t_{i-1}}$ or $z \in \mu_{t_{i-1}}$; i. e., $\mu(x) \ge t_{i-1}$ or $t_i = \mu(z) \ge t_{i-1}$. This is a contradiction and the proof is complete.

THEOREM 4. *Let $\{\mu_i\}$ be any chain of fuzzy prime ideals of M. Then $\cup\mu_i$ is a fuzzy prime ideal of M.*

Proof. It is easily proved that $\cup\mu_i$ is a fuzzy ideal of M. Since each μ_i is prime, it follows from Lemma 1 that for all i,

(i) $1 \in Im(\mu_i)$,

(ii) the ideal $M_{\mu_i} = \{x \in M | \mu_i(x) = 1\}$ is prime,

(iii) there exists $t_i \in [0,1)$ such that $\mu_i(x) = t_i$ for all $x \in M - M_{\mu_i}$.

As $\mu_1 \subseteq \mu_2 \subseteq ... \subseteq \mu_n \subseteq ...$(say); we have $M_{\mu_1} \subseteq M_{\mu_2} \subseteq ... \subseteq M_{\mu_n} \subseteq ...$, and hence $\cup M_{\mu_i}$ is a prime ideal of M. Now let σ and ρ be any two fuzzy ideals of M such that $\sigma\Gamma\rho \subseteq \cup\mu_i$. Assume that $\cup\mu_i$ is not fuzzy prime. Then there exist $x, y \in M$ such that $\sigma(x) > (\cup\mu_i)(x)$ and $\rho(y) > (\cup\mu_i)(y)$. Therefore $(\cup\mu_i)(x) \neq 1$ and $(\cup\mu_i)(y) \neq 1$, so that $x, y \notin \cup M_{\mu_i}$. Since $\cup M_{\mu_i}$ is prime, it follows from Proposition 1 that $x\Gamma M\Gamma y \not\subseteq \cup M_{\mu_i}$. Hence $(\cup\mu_i)(x) = (\cup\mu_i)(y) = (\cup\mu_i)(x\alpha z\beta y) = \sup_i t_i$ for all $z \in M$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}(\sigma\Gamma\rho)(x\alpha z\beta y) &\geq \min\{\sigma(x), \rho(y)\} \\ &> \min\{(\cup\mu_i)(x), (\cup\mu_i)(y)\} \\ &= (\cup\mu_i)(x\alpha z\beta y)\end{aligned}$$

for all $z \in M$ and $\alpha, \beta \in \Gamma$. This is a contradiction, and the proof is complete.

DEFINITION 9 ([12]). Let $\theta : M \to M'$ be any function. A fuzzy set μ in M is called θ-invariant if $\theta(x) = \theta(y)$ implies $\mu(x) = \mu(y)$, where $x, y \in M$.

LEMMA 2. *Let $\theta : M \to M'$ be an onto Γ-homomorphism and let μ be any θ-invariant fuzzy ideal of M such that $Im(\mu) = \{t_0, t_1, ..., t_m\}$ where $t_0 > t_1 > ... > t_m$. If the chain of level ideals of μ is $\mu_{t_0} \subset \mu_{t_1} \subset ... \subset \mu_{t_m} = M$, then the chain of level ideals of $\theta(\mu)$ is given by $\theta(\mu_{t_0}) \subseteq \theta(\mu_{t_1}) \subseteq ... \subseteq \theta(\mu_{t_m}) = M'$.*

Proof. Clearly $Im(\theta(\mu)) \subseteq Im(\mu)$. If $(\theta(\mu))_{t_i} = \theta(\mu_{t_i})$, then the chain of level ideals of $\theta(\mu)$ is given by $\theta(\mu_{t_0}) \subseteq \theta(\mu_{t_1}) \subseteq ... \subseteq \theta(\mu_{t_m}) = M'$. Hence we need only to prove that $(\theta(\mu))_{t_i} = \theta(\mu_{t_i})$. Let $y \in (\theta(\mu))_{t_i}$. Then $t_i \leq \theta(\mu)(y) = \sup_{z \in \theta^{-1}(y)} \mu(z)$, and so $\mu(x) \geq t_i$ for some $x \in \theta^{-1}(y)$. Hence $x \in \mu_{t_i}$, and $y = \theta(x) \in \theta(\mu_{t_i})$, showing that $(\theta(\mu))_{t_i} \subseteq \theta(\mu_{t_i})$. To prove the reverse inclusion, let $y \in \theta(\mu_{t_i})$. Then there exists $x \in \mu_{t_i}$ such that $y = \theta(x)$. Thus $\mu(x) \geq t_i$, and $\theta(\mu)(y) = \sup_{z \in \theta^{-1}(y)} \mu(z) \geq \mu(x) \geq t_i$, which means that $y \in (\theta(\mu))_{t_i}$. This completes the proof.

THEOREM 5. *Let $\theta : M \to M'$ be an onto Γ-homomorphism. If μ is a θ-invariant fuzzy prime ideal of M, then $\theta(\mu)$ is a fuzzy prime ideal of M'.*

Proof. Since μ is a fuzzy prime ideal of M, it follows from Lemma 1 that (i)$1 \in Im(\mu)$, (ii) the ideal $M_\mu = \{x \in M | \mu(x) = 1\}$ is prime and (iii) there exists $t \in [0, 1)$ such that $\mu(x) = t$ for all $x \in M - M_\mu$. Now we show that $ker(\theta) \subseteq M_\mu$. Let $x \in ker(\theta)$. Then $\theta(x) = 0_{M'} = \theta(0_M)$. As μ is θ-invariant and $\mu(0_M) = 1$, we have that $\mu(x) = \mu(0_M) = 1$. Hence $x \in M_\mu$, showing that $ker(\theta) \subseteq M_\mu$. In view of the fact that M_μ is a prime ideal and $ker(\theta) \subseteq M_\mu$, we have that $\theta(M_\mu)$ is a prime ideal of M'. Since $M_\mu \subset M$, it follows from Lemma 2 that $\theta(M_\mu) \subset \theta(M) = M'$ (the inclusion is strict, because μ is θ-invariant). Now $\theta(\mu)(0_{M'}) = \sup_{x \in \theta^{-1}(0_{M'})} \mu(x) \geq \mu(0_M) = 1$, and hence $1 \in Im(\theta(\mu))$. Therefore the result follows from Lemma 1.

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Department of Mathematics Education
Gyeongsang National University
Chinju 660-701, Korea

Department of Mathematics Education
Kyungnam University
Masan 631-701, Korea