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ON ANALYTIC FUNCTIONS WITH CERTAIN DIFFERENTIAL INEQUALITY*

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I. Introduction

Miller [4] and Lewandowski, Miller and Zlotkiewicz [2] showed that an analytic function satisfying a differential inequality of a certain type is necessarily a Caratheodory function.

This result has wide applications in the theory of univalent functions but it is not sharp for the special cases.It is the purpose of the present paper to give sharp results for the special cases.

We shall say that $p(z)$ belongs to the class P, if $p(z)$ is analytic in the unit disk $D = \{z : |z| < 1\}, p(0) = 1$ and $Rep(z) > 0$ in D. If $p(z)$ is in P, we say $p(z)$ a Caratheodory function.

Miller, Mocanu and Reade [3] proved the following very interesting and usefull result:

Theorem A. *If $p(z)$ is analytic in D, $p(0) = 1$, $p(z) \neq 0$ in D and if α is real, then for z in D*

$$Re[p(z) + \alpha \frac{zp'(z)}{p(z)}] > 0 \tag{1}$$

$$\Longrightarrow Rep(z) > 0;$$

that is, $p(z)$ is a Caratheodory function.

Miller [4] replaced the differential inequality in (1) by a much more general condition which will still imply that $p(z)$ is a Caratheodory function.

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Applying Miller's result [4, theorem 1 and it's remark], we can easily obtain the following results:

(2) $$Re[p(z)+\alpha zp'(z)]>0, \quad \text{with} \quad \alpha \geq 0 \quad \text{in } D$$

$$\Longrightarrow Rep(z)>0 \quad \text{in } D,$$

(3) $$P(z)\neq 0 \quad \text{and} \quad Re[p(z)-\frac{zp'(z)}{p(z)^2}]>0 \quad \text{in } D$$

$$\Longrightarrow ReP(z)>0 \quad \text{in } D,$$

(4) $$Re[p(z)^2+2zp'(z)p(z)]>0 \quad \text{in } D$$

$$\Longrightarrow Rep(z)>0 \quad \text{in } D.$$

Putting $\alpha=1$ in (1) and (2), we obtain the following results respectively;
if $p(z)$ is analytic in D, $p(0)=1$ and

(1') $$p(z)\neq o \quad \text{and} \quad Re[p(z)+\frac{zp'(z)}{p(z)}]>0 \quad \text{in } D.$$

$$\Longrightarrow Rep(z)>0 \quad \text{in } D,$$

and

(2') $$Re[p(z)+zp'(z)]>0 \quad \text{in } D$$

$$\Longrightarrow Rep(z)>0 \quad \text{in } D.$$

In this paper, we need the following lemma.

Lemma 1. *If $p(z) \in P$, then we have*

$$p(z) \prec \frac{1+z}{1-z} \qquad \text{in } D$$

where $\prec$ means subordination A proof can be found in [1, Vol. 1, Theorem 7, p. 84 and 87].

2. Main Theorems.

Theorem 1. *Let $p(z)$ be analytic in D, $p(0) = 1$ and suppose that*

$$Re[p(z) + zp'(z)] > 0 \qquad \text{in } D.$$

Then we have

$$Re p(z) > \log\frac{4}{e} > 0 \qquad \text{in } D.$$

Proof. Let us put $g(z) = zp(z)$. Then we have

$$Re g'(z) = Re[p(z) + zp'(z)] > 0 \qquad \text{in } D$$

and

$$g'(0) = 1.$$

From Lemma 1, we have

$$g'(z) \prec \frac{1+z}{1-z} \qquad \text{in } D. \tag{5}$$

Now, we have

$$\begin{aligned} p(z) = \frac{g(z)}{z} &= \frac{1}{z}\int_0^z g'(s)ds \\ &= \frac{1}{z}\int_0^r g'(te^{i\phi})e^{i\phi}dt \\ &= \frac{1}{r}\int_0^r g'(te^{i\phi})dt \end{aligned}$$

where $z = re^{i\phi}, 0 < r < 1, s = te^{i\phi}$ and $0 \leq t \leq r$.

Therefore, from (5) and [1, Theorem 7, p.84], we have

$$\begin{aligned} Rep(z) &= \frac{1}{r}\int_0^r Reg'(te^{i\phi})dt \\ &\geq \frac{1}{r}\int_0^r \frac{1-t}{1+t}dt \\ &= \frac{1}{r}[-r + 2log(1+r)] \\ &= 2log(1+r)^{\frac{1}{r}} - 1 \\ &> log4 - 1 = log\frac{4}{e} \end{aligned}$$

for $0 < r < 1$. This shows that

$$Rep(z) > log\frac{4}{e} \qquad \text{in } D.$$

This completes our proof and the result is trivially sharp.

Applying the same method as in the proof of Theorem 1, we have the following corollaries.

Corollary 1. *Let $p(z)$ be analytic in D, $p(0) = 1$ and suppose that*

$$Re[p(z)^2 + 2zp'(z)p(z)] > 0 \qquad \textit{in } D.$$

Then we have

$$[Rep(z)]^2 > log\frac{4}{e} \qquad \textit{in } D.$$

Proof. Putting $g(z) = zp(z)^2$, then from Theorem 1, we easily have

$$\begin{aligned} [Rep(z)]^2 &\geq [Rep(z)]^2 - [Imp(z)]^2 \\ &> log\frac{4}{e} \qquad \text{in } D. \end{aligned}$$

Corollary 2. *Let $p(z)$ be analytic in D, $p(0) = 1$, $p(z) \neq 0$ in D and*

$$Re[p(z) - \frac{zp'(z)}{p(z)^2}] > 0 \qquad \textit{in } D.$$

Then we have

$$0 < Rep(z) < [log\frac{4}{e}]^{-1} \qquad in\ D.$$

Proof. Putting $g(z) = z/p(z)$ in the proof of Theorem 1 and $p(z) = u + iv$ where u and v are real, then from Theorem 1, we have

$$Re\frac{1}{p(z)} = \frac{u}{u^2+v^2} > log\frac{4}{e} \qquad \text{in } D.$$

This shows that

$$u = Rep(z) > 0 \qquad \text{in } D,$$

and therefore, we have

$$0 < Rep(z) < [log\frac{4}{e}]^{-1} \qquad \text{in } D$$

Corollary 3. *Let $p(z)$ be analytic in D, $p(0) = 1$, $p(z) \neq 0$ and suppose that*

$$1 + log|p(z)| + Re\frac{zp'(z)}{p(z)} > 0 \qquad in\ D.$$

Then we have

$$|p(z)| > \frac{4}{e^2} \qquad in\ D.$$

Proof. Putting $g(z) = zlog[ep(z)]$ in the proof of Theorem 1, we easily have

$$Relog[ep(z)] = loge|p(z)| > log\frac{4}{e} \qquad \text{in } D.$$

This shows that $|p(z)| > \frac{4}{e^2}$ in D. This completes our proof.

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