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ON THE EQUIVALENCES OF TWO TYPE CONDITIONS CONCERNING FUNCTIONS

Gue Myung Lee, Do Sang Kim and Kuk Hyeon Son

1. Introduction.

Recently, Hanson [4], and Kaul and Kaur [7] defined a new class of functions, called invex, pseudo invex and quasi invex. They have established Kuhn-Tucker sufficient optimality criteria and duality theorems for nonlinear programming problems involving these functions. Jeyakumar [6] defined ρ-invex, ρ-pseudo invex and ρ-quasi invex functions which are generalized forms of invex, pseudo invex, and quasi invex functions respectively. His functions also generalize Vial's ρ-convex functions [8]. On the other hand, Hanson and Mond [5], and Egudo and Mond [3] introduced a class of functions, called F-convex, F-pseudoconvex and F-quasiconvex. They also have obtained Kuhn-Tucker sufficient optimality criteria and duality theorems for nonlinear programming problems concerning these functions. By using the fact that invex functions can be characterized as functions whose stationary points are global minima, Craven and Glover [2], and Ben-Israel and Mond [1] showed that the invex condition is equivalent to the pseudo invex condition, and that the invex condition is equivalent to the F-convex condition. Our object of this brief paper is to show that the quasi invex condition is equivalent to F-quasiconvex conditions, by using a simple method. Moreover, we give conditions equivalent to ρ-invex, ρ-pseudo invex and ρ-quasi invex conditions.

2. Equivalences

Now we define invexity type functions and F-convexity type functions.

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DEFINITION 2.1 ([4],[7]). *Let f be a differentiable numerical function defined on a set $C \subset R^n$.*

(1) f is said to be invex w.r.t. η if there exists a vector function $\eta(x,u)$ defined on $C \times C$ such that $f(x) - f(u) \geq \eta^t(x,u)\nabla f(u)$ for all $x, u \in C$.

(2) f is said to be pseudo invex w.r.t. η if there exists a vector function $\eta(x,u)$ defined on $C \times C$ such that $\eta^t(x,u)\nabla f(u) \geq 0 \Rightarrow f(x) \geq f(u)$ for all $x, u \in C$.

(3) f is said to be quasi invex w.r.t. η if there exists a vector function $\eta(x,u)$ such that $f(x) \leq f(u) \Rightarrow \eta^t(x,u)\nabla f(u) \leq 0$ for all $x, u \in C$.

DEFINITION 2.2. *A functional F is said to be sublinear on R^n if*

$$F(x+y) \leq F(x) + F(y) \ \textit{for all} \ x, y \in R^n.$$
$$F(\alpha x) = \alpha F(x) \ \textit{for all} \ \alpha \in R, \ \alpha \geq 0 \ \textit{and} \ x \in R^n.$$

DEFINITION 2.3 ([3],[5]). *Let f be a differentiable numerical function defined on a set $C \subset R^n$.*

(1) f is said to be F-convex if there exist sublinear functionals $F_{x,u}$ such that $f(x) - f(u) \geq F_{x,u}[\nabla f(u)]$ for all $x, u \in C$.

(2) f is said to be F-pseudoconvex if there exist sublinear functionals $F_{x,u}$ such that $F_{x,u}[\nabla f(u)] \geq 0 \Rightarrow f(x) \geq f(u)$ for all $x, u \in C$.

(3) f is said to be F-quasiconvex if there exist sublinear functionals $F_{x,u}$ such that $f(x) \leq f(u) \Rightarrow F_{x,u}[\nabla f(u)] \leq 0$ forall $x, u \in C$.

Now we show the equivalences among the above functions.

THEOREM 2.1. Let f be a differentiable numerical function defined on a set $C \subset R^n$. Then the following statements are equivalent.

(a) there exists a vector function $\eta(x,u)$ defined on $C \times C$ such that f is quasi invex w.r.t. η.

(b) there exist sublinear functionals $F_{x,u}$ such that f is F-quasiconvex.

PROOF.

(a) $\Rightarrow$ (b) : By assumption, $f(x) \leq f(u) \Rightarrow \eta^t(x,u)\nabla f(u) \leq 0$ for all $x, u \in C$. Let $\eta^t(x,u)z = F_{x,u}(z)$ for all $x, u \in C$ and $z \in R^n$. Then $F_{x,u}$ is sublinear and hence f is F-quasiconvex.

(b) $\Rightarrow$ (a) : By assumption, $f(x) \leq f(u) \Rightarrow F_{x,u}[\nabla f(u)] \leq 0$ for all $x, u \in C$. If $\nabla f(u) = 0$, *let* $\eta(x,u) = 0$. If $\nabla f(u) \neq 0$, let

$\eta(x,u) = \frac{F_{x,u}[\nabla f(u)]}{\nabla f(u)^t \nabla f(u)} \nabla f(u)$ and hence $\eta^t(x,u)\nabla f(u) = F_{x,u}[\nabla f(u)]$.

Thus we obtain $f(x) \leq f(u) \Rightarrow \eta^t(x,u)\nabla f(u) \leq 0$ for all $x, u \in C$.

The following theorems are established in a manner similar to Theorem 2.1.

THEOREM 2.2. *Let f be a differentiable numerical function defined on a set $C \subset R^n$. Then the following statements are equivalent.*

(a) there exists a vector function $\eta(x,u)$ defined on $C \times C$ such that f is invex w.r.t. η.

(b) there exist sublinear functionals $F_{x,u}$ such that f is F-convex.

THEOREM 2.3. *Let f be a differentiable numerical function defined on a set $C \subset R^n$. then the following statements are equivalent.*

(a) there exists a vector function $\eta(x,u)$ defined on $C \times C$ such that f is pseudo invex w.r.t. η.

(b) there exist sublinear functionals $F_{x,u}$ such that f is F-pseudoconvex.

The following definitions were suggested by Jeyakumar [6].

DEFINITION 2.4. *Let f be a differentiable numerical function defined on a set $C \subset R^n$.*

(1) f is said to be ρ-invex w.r.t. η and θ if there exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that $f(x) - f(u) \geq \eta^t(x,u)\nabla f(u) + \rho\|\theta(x,u)\|^2$ for all $x, u \in C$.

(2) f is said to be ρ-pseudo invex w.r.t. η and θ if there exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that $\eta^t(x,u)\nabla f(u) \geq -\rho\|\theta(x,u)\|^2 \Rightarrow f(x) \geq f(u)$ for all $x, u \in C$.

(3) f is said to be ρ-quasi invex w.r.t. η and θ if there exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that $f(x) \leq f(u) \Rightarrow \eta^t(x,u)\nabla f(u) \leq -\rho\|\theta(x,u)\|^2$ for all $x, u \in C$.

It is easy for us to obtain the following theorems.

THEOREM 2.4. *Let f be a differentiable numerical function defined on a set $C \subset R^n$ Then the following statements are equivalent.*

(a) There exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that f is ρ-invex w.r.t. η and θ.

(b) There exist sublinear functionals $F_{x,u}$ such that
$f(x) - f(u) \geq F_{x,u}[\nabla f(u)] + \rho\|\theta(x,u)\|^2$ *for all* $x, u \in C$.

THEOREM 2.5. *Let f be a differentiable numerical function defined on a set $C \subset R^n$. Then the following statements are equivalent.*

(a) There exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that f is ρ-pseudo invex w.r.t. η and θ.

(b) There exist sublinear functionals $F_{x,u}$ such that
$F_{x,u}[\nabla f(u)] \geq -\rho\|\theta(x,u)\|^2 \Rightarrow f(x) \geq f(u)$*for all* $x, u \in C$.

THEOREM 2.6. *Let f is be a differentiable numerical function defined on a set $C \subset R^n$. Then the following statements are equivalent.*

(a) There exist vector functions $\eta(x,u)$ and $\theta(x,u)$ defined on $C \times C$, and a real number ρ such that f is ρ-quasi invex w.r.t. η and θ.

(b) There exist a sublinear functionals $F_{x,u}$ such that
$f(x) \leq f(u) \Rightarrow F_{x,u}[\nabla f(u)] \leq -\rho\|\theta(x,u)\|^2$ *for all* $x, u \in C$.

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Department of National Sciences
Pusan National University of Technology
Pusan 608-739, Korea

Department of Applied Mathematics
National Fisheries University of Pusan
Pusan 608-737, Korea

Department of Mathematics
Dongseo University
Pusan 616-010, Korea.