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ON CLASSES OF MULTIVALENT FUNCTIONS DEFINED BY CERTAIN DIFFERENTIAL OPERATOR

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1. Introduction

Let A_p denote the class of functions

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in N = \{1, 2, 3, \cdots\}) \tag{1.1}$$

which are analytic in the unit disk $U = \{z | \ |z| < 1\}$. For $0 \leq \alpha < 1$, we denote by $S_p^*(\alpha)$ and $K_p(\alpha)$ the classes of p-valent starlike functions of order α and p -valent convex functions of order α, respectively [1].

For $f \in A_p$, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = z \left(\frac{f(z)}{p} \right)' \tag{1.3}$$

and

$$D^n f(z) = D\left(D^{n-1} f(z)\right) \quad (n \in N). \tag{1.4}$$

Now we introduce the following classes by using the differential operator D^n.

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Definition. A function $f \in A_p$ is said to be p-valently n- starlike functions of order α if f satisfies the condition

$$Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1, \ z \in U). \tag{1.5}$$

We denote by $S_{n,p}(\alpha)$ the class of p-valently n-starlike functions of order α. We note that $S_{0,p}(\alpha) = S_p^*(\alpha)$ and $S_{1,p}(\alpha) = K_p(\alpha)$. For $p = 1$, the class $S_{n,1}(\alpha)$ is considered by Salagean [7].

In this paper, we give certain inequalities for $f \in A_p$ which satisfies the condition

$$Re\left\{\frac{D^n f(z)}{z^p}\right\} > \alpha \quad (0 \leq \alpha < 1, \ z \in U) \tag{1.6}$$

and for the following integral (1.7) of functions satisfying (1.5)

$$F(z) = \frac{p+c}{z^c}\int_0^z u^{c-1}f(u)du \quad (c > -p). \tag{1.7}$$

These inequalities include or improve several results given by Bernardi [2], Jack [3], Libera [4], Obradovic [5,6] and Strohacker [8].

2. Main results

We need the following lemma due to Jack [3] for the proofs of the comming results.

Lemma 1. Let w be a nonconstant and analytic function in $|z| < r < 1$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r$ at z_0, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number and $k \geq 1$.

Theoem 1. Let $f \in S_{n,p}(\alpha)$ and let

$$F(z) = \frac{p+c}{z^c}\int_0^z u^{c-1}f(u)du \quad (c > -p). \tag{2.1}$$

Then

$$\text{(2.2)} \qquad Re\left\{\frac{D^{n+1}F(z)}{D^nF(z)}\right\} > \beta(\alpha, p, c),$$

where $c \geq 2p(1-\alpha)-(p+1)$ and

(2.3)

$$\beta(\alpha, p, c) = \frac{-(2c-2\alpha p+1)+\sqrt{(2c-2\alpha p+1)^2+8p(2\alpha c+1)}}{4p}.$$

Proof. Suppose that $f \in S_{n,p}(\alpha)$ satisfies the conditions in the theorem and write

$$\text{(2.4)} \qquad \frac{D^{n+1}F(z)}{D^nF(z)} = \frac{1+(2\beta-1)w(z)}{1+w(z)},$$

where $\beta = \beta(\alpha, p, c)$. Then $w(z)$ is analytic, $w(0) = 0$ and $w(z) \neq -1$ in U. Using the identity

$$\text{(2.5)} \qquad (p+c)D^nf(z) = cD^nF(z) + pD^{n+1}F(z),$$

the equation (2.4) may be written as

$$\text{(2.6)} \qquad \frac{D^nf(z)}{D^nF(z)} = \frac{c(1+w(z))+p(1+(2\beta-1)w(z))}{(p+c)(1+w(z))}.$$

Differentiating (2.6) logarithmically, we obtain

(2.7)

$$\frac{D^{n+1}f(z)}{D^nf(z)} = \frac{1+(2\beta-1)w(z)}{1+w(z)} - \frac{2(1-\beta)zw'(z)}{(1+w(z))(c+p+(c+p(2\beta-1))w(z))}.$$

We claim thet $|w(z)| < 1$. For otherwise, by Lemma 1, there exists $z_0 \in U$ such that

$$z_0 w'(z_0) = kw(z_0), \tag{2.8}$$

where $|w(z_0)| = 1$ and $k \geq 1$. Writing $w(z_0) = u + iv$, the equation (2.7) in conjuction with (2.8) yields

(2.9)

$$\begin{aligned}
&Re\left\{\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - \alpha\right\} \\
&= \beta - \alpha - 2(1-\beta)kRe\left\{\frac{u+iv}{(1+u+iv)(c+p+(c+p(2\beta-1))(u+iv)}\right\} \\
&= \beta - \alpha - 2(1-\beta)k\left\{\frac{(1+u+iv)(a+bu-ibv)}{2(1+u)((a+bu)^2+b^2v^2)}\right\} \\
&= \beta - \alpha - \frac{(1-\beta)k(a+b)}{a^2+2abu+b^2},
\end{aligned}$$

where $a = c + p$ and $b = c + p(2\beta - 1)$. Put

$$g(u) = \frac{(a+b)}{a^2+2abu+b^2}. \tag{2.10}$$

The condition $c \geq 2p(1-\alpha) - (p+1)$ and the definition of $\beta(\alpha, p, c)$ imply $b \geq 0$ and $\beta < 1$. Then $g(u)$ is decreasing and thus $\frac{1}{a+b} = g(1) \leq g(u)$. We have, from (2.9) and $k \geq 1$, that

$$\begin{aligned}
Re\left\{\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - \alpha\right\} &\leq \beta - \alpha - \frac{(1-\beta)}{a+b} \\
&= 2p\beta^2 + (2c - 2\alpha p + 1)\beta - (2\alpha c + 1) = 0,
\end{aligned} \tag{2.11}$$

since β is a root of the polynomial

$$2px^2 + (2c - 2\alpha p + 1)x - (2\alpha c + 1) = 0. \tag{2.12}$$

This contadicts the assumption $f \in S_{n,p}(\alpha)$ and so the proof is completed.

From (2.1), for $p = 1$, we note that

$$\frac{zF'(z)}{F(z)} = \frac{DF(z)}{F(z)} = -c + \frac{z^c f(z)}{\int_0^z u^{c-1} f(u)du}. \tag{2.13}$$

Taking $p = 1$ and $n = 0$ in Theorem 1, we obtain the following corollary which was proved by Obradovic [6].

Corollary 1. Let $f \in S_1^*(\alpha)$ and let $c > max\{-1, -2\alpha\}$. Then we have

(2.14)

$$Re\left\{\frac{z^c f(z)}{\int_0^z u^{c-1} f(u)du}\right\} > \frac{2c + 2\alpha - 1 + \sqrt{(2c + 2\alpha - 1)^2 + 8(c+1)}}{4}$$
$(z \in U)$.

Theorem 2. Let $f \in S_{n,p}(\alpha)$ and $\gamma \geq 1$. Then

$$Re\left\{\frac{D^n f(z)}{z^p}\right\}^{\frac{1}{2p(1-\alpha)\gamma}} > \frac{\gamma}{\gamma + 1} \quad (z \in U). \tag{2.15}$$

Proof. Let $\beta = \frac{\gamma}{1+\gamma}$ and let $w(z)$ be an analytic function such that

$$\left\{\frac{D^n f(z)}{z^p}\right\}^{\frac{1}{2p(1-\alpha)\gamma}} = \frac{1 + (2\beta - 1)w(z)}{1 + w(z)}. \tag{2.16}$$

Then $w(0) = 0$ and $w(z) \neq -1$ in U. The theorem will follow if we can show that $|w(z)| < 1$ in U. Now by differentiating (2.16) logarithmically, we get

$$\frac{D^{n+1}f(z)}{D^n f(z)} = 1 - \frac{4(1-\alpha)\gamma(1-\beta)zw'(z)}{(1+w(z))(1+(2\beta-1)w(z))}. \tag{2.17}$$

If $|w(z)| \not< 1$ in U, by Lemma 1, there exists $z_0 \in U$ such that $z_0 w'(z_0) = kw(z_0)$, where $|w(z_0)| = 1$ and $k \geq 1$. Let $w(z_0) = u + iv$. Then

$$\begin{aligned} &Re\left\{\frac{z_0 w'(z_0)}{(1+w(z_0))(1+(2\beta-1)w(z_0))}\right\} \\ &= \frac{\beta k}{2(2\beta^2 - 2\beta + 1 + (2\beta-1)u)}. \end{aligned} \tag{2.18}$$

Put

$$g(u) = \frac{1}{2\beta^2 - 2\beta + 1 + (2\beta - 1)u}. \tag{2.19}$$

Since $\gamma \geq \frac{1}{2}$ implies $g(u)$ is an decreasing function of u, $\frac{1}{2\beta^2} = g(1) \leq g(u)$. Applying (2.17) and (2.18), we obtain

(2.20)

$$\begin{aligned} Re\left\{\frac{D^{n+1}f(z_0)}{D^n f(z_0)} - \alpha\right\} &= 1 - \alpha - \frac{4(1-\alpha)\gamma(1-\beta)\beta k}{2(2\beta^2 - 2\beta + 1 + (2\beta-1)u)} \\ &= 1 - \alpha - 2(1-\alpha)\gamma(1-\beta)\beta k g(u) \\ &\leq 1 - \alpha - \frac{(1-\alpha)\gamma(1-\beta)}{\beta} = 0, \end{aligned}$$

which contradicts the assumption. Thus the theorem is proved.

Putting $p = 1$, $\gamma = \frac{1}{1-\alpha}$ and replacing n by $n+1$ in Theorem 2, we obtain the following corollary.

Corollary 2. Let $f \in S_{n+1,1}(\alpha)$. Then

$$(2.21) \qquad Re\left\{\frac{D^{n+1}f(z)}{z}\right\}^{\frac{1}{2(1-\alpha)}} > \frac{1}{2-\alpha} \ (z \in U).$$

Under the condition of Corollary 2, taking $n = 0$ and $\alpha = 0$, we have the known result of Strohacker [8], that is, $f \in K_1(0)$ implies $Re\{\sqrt{f'(z)}\} > \frac{1}{2}$.

By considering $p = 1$, $n = 0$ and $\gamma = 1$ in Theorem 2, we have the following result of Jack [3].

Corollary 3. Let $f \in S_1^*(\alpha)$. Then

$$(2.22) \qquad Re\left\{\frac{f(z)}{z}\right\}^{\frac{1}{2(1-\alpha)}} > \frac{1}{2} \ (z \in U).$$

Recently Obradovic [5] proved the following result which can be derived from Theorem 2 by taking $p = 1$, $n = 0$ and $\gamma = \frac{1}{2(1-\alpha)}$.

Corollary 4. Let $f \in S_1^*(\alpha)$ Then

$$(2.23) \qquad Re\left\{\frac{f(z)}{z}\right\} > \frac{1}{3-2\alpha} \ (z \in U).$$

Theorem 3. Let $Re\ c > -p$, $0 \le \alpha < 1$ and $f \in A_p$. If

$$(2.24) \qquad Re\left\{\frac{D^{n+1}f(z)}{z^p}\right\} > \alpha \ (z \in U),$$

then

$$(2.25) \qquad Re\left\{\frac{D^{n+1}F(z)}{z^p}\right\} > \beta(\alpha, p, c) \ (z \in U),$$

where

$$\beta(\alpha, p, c) = \frac{\alpha + \frac{1}{2} Re\left\{\frac{1}{c+p}\right\}}{1 + \frac{1}{2} Re\left\{\frac{1}{c+p}\right\}} \tag{2.26}$$

and $F(z)$ is defined as in (2.1).

Proof. As Theorem 1, we assume that the function f satisfies the conditions in the theorem and write

$$\frac{D^{n+1}F(z)}{z^p} = \frac{1 + (2\beta - 1)w(z)}{1 + w(z)}. \tag{2.27}$$

where $\beta = \beta(\alpha, p, c)$. Then $w(z)$ is analytic, $w(0) = 0$ and $w(z) \neq -1$ in U. It is sufficient to show that $|w(z)| < 1$ for $z \in U$. From (2.5) and (2.27), we have

$$\frac{D^{n+1}f(z)}{z^p} = \frac{1 + (2\beta - 1)w(z)}{1 + w(z)} - \frac{2(1 - \beta)zw'(z)}{(p + c)(1 + w(z))^2}. \tag{2.28}$$

$|w(z)| \not< 1$, there exists $z_0 \in U$ so that $|w(z)| \leq |w(z_0)| = 1$ for $z \in U$. Then, by Lemma 1, there exists $k \geq 1$ such that

$$z_0 w'(z_0) = k w(z_0). \tag{2.29}$$

Let $w(z_0) = u + \imath v$ so that

$$Re\left\{\frac{z_0 w'(z_0)}{(1 + w(z_0))^2}\right\} = \frac{k}{2(1 + u)} \tag{2.30}$$

and take the the real part of (2.28). Then we obtain

(2.31)
$$\begin{aligned} &Re\left\{\frac{D^{n+1}f(z_0)}{z_0^p}-\alpha\right\} \\ &=\beta-\alpha-\frac{(1-\beta)k}{(1+u)}Re\left\{\frac{1}{c+p}\right\} \\ &\le \beta-\alpha-\frac{(1-\beta)}{2}Re\left\{\frac{1}{c+p}\right\} \\ &=\beta\left(1+\frac{1}{2}Re\left\{\frac{1}{c+p}\right\}\right)-\alpha-\frac{1}{2}Re\left\{\frac{1}{c+p}\right\} \\ &=0, \end{aligned}$$

which contradicts the assumption. So $|w(z)| < 1$ for $z \in U$. This completes the proof of theorem.

Remarks. (i) Taking $n = \alpha = 0$ and $p = 1$ in Theorem 3, we have Bernardi's results [2]: If $Re\{f'(z)\} > 0$, then $Re\{F'(z)\} > 0$.

(ii) Putting $n = \alpha = 0$, $p = 1$ and $c = 1$ in Theorem 3, we have a result of Libera [4].

For $p = 1$, Obracdovic [5,6] recently gave the following two results which can also be obtained from Theorem 3 by $n = c = \alpha = 0$ and $n = -1$, respectively.

Corollary 5. Let $f \in A_1$. Then $Re\{f'(z)\} > 0$ implies $Re\{\frac{f(z)}{z}\} > \frac{1}{3}$ $(z \in U)$.

Corollary 6. Let $f \in A_1$, $0 \le \alpha < 1$ and $c > -1$. Then $Re\{\frac{f(z)}{z}\} > \alpha$ implies

$$Re\left\{\frac{c+1}{z^{c+1}}\int_0^z u^{c-1}f(u)du\right\} > \alpha+\frac{1-\alpha}{3+2c} \quad (z \in U). \tag{2.32}$$

We state the following theorem which is proved by a similar method.

Theorem 4. Let $f \in A_p$ and

$$Re\left\{\frac{D^{n+2}f(z)}{z^p}\right\} > \alpha \quad (0 \le \alpha < 1, z \in U). \tag{2.33}$$

Then

$$Re\left\{\frac{D^{n+1}f(z)}{z^p}\right\} > \frac{2\alpha p+1}{2p+1} \quad (z \in U). \tag{2.34}$$

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