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GLOBAL EXISTENCE OF HOLOMORPHIC SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH COMPLEX PARAMETERS–II

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1. Introduction and the Main Theorem

As early as in 1956 L. Ehrenpreis[1] discoursed on an application of the sheaf theory to differential equations and gave a criterion for the existence of global solutions of differential equations $Tf = g$ in a domain D when the existence of local solutions are assured. The first author[2] applied Ehrenpreis' method to linear ordinary differential equations with meromorphic coefficients and gave a necessary and sufficient condition for the global existence in the meromorphic category. In the holomorphic category the condition is that D is either simply connected or doubly connected without non trivial global single-valued holomorphic homogeneous solutions.

H. Suzuki[11] stated that the global existence of differential equation $\partial u/\partial x_1 = f$ with complex parameter depends on the Steinness of the sets of cuts over the parameter space. The first author and Y. Mori[7] connected [2] and [11] and discussed the global existence of more general ordinary differential equations with complex parameters. The authors[8] generalized those to the case of Stein parameter spaces.

Concerning partial differential equations, the first author[3]-[6] discussed several concrete cases. In the former paper K. H. Shon[10] reported promptly the main theorem and showed sheaf-theoretically a route of its proof. The main of this series is to give complete analytical foundations to it. In the previous paper[9], we gave a necessary and

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sufficient condition for the global existence in case of polycylindrical domains as follows:

Let m be a positive integer and, for each integer j with $1 \leq j \leq m$, D_j be a domain in the complex plane $\mathbb{C}$. We put

$$D = D_1 \times D_2 \times \cdots \times D_m. \tag{1.1}$$

The set D is a domain of holomorphy in the complex m-space $\mathbb{C}^m$. Let M be a Stein manifold. We use the manifold M as a parameter space, denote by r a point of M and regard it as a complex parameter. We consider the product manifold $D \times M$.

For each i with $1 \leq i \leq m$, let $a^i = (a^i_{jk}(z, r))$ be a square matrix of degree m whose each (j, k) element $a^i_{jk}(z, r)$ is holomorphic function in $D \times M$. For each i with $1 \leq i \leq m$, let T_i be a differential operator defined by

$$T_i = \frac{\partial}{\partial z_i} + a^i. \tag{1.2}$$

Let $\mathcal{O}_{D\times M}$ be the sheaf of germs of all holomorphic function on $D \times M$. Then each differential operator T_i difines a sheaf homomorphism

$$T_i : (\mathcal{O}_{D\times M})^m \longrightarrow (\mathcal{O}_{D\times M})^m \tag{1.3}$$

as

$$T_i u = \begin{pmatrix} \frac{\partial u_1}{\partial z_1} + \sum_{k=1}^m a^i_{1k}(z, r) u_k \\ \frac{\partial u_2}{\partial z_2} + \sum_{k=1}^m a^i_{2k}(z, r) u_k \\ \vdots \\ \frac{\partial u_m}{\partial z_m} + \sum_{k=1}^m a^i_{mk}(z, r) u_k \end{pmatrix} \tag{1.4}$$

for any germ

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$$

of $(\mathcal{O}_{D\times M})^m$. We put

$$T = T_1 T_2 \cdots T_m. \tag{1.5}$$

Let

$$(1.6) \qquad H(D \times M) = H^0(D \times M, (\mathcal{O}_{D\times M})^m)$$

be the set of global sections of $(\mathcal{O}_{D\times M})^m$ over $D \times M$. $H^0(D \times M, \mathcal{O}_{D\times M})$ is the $\mathbb{C}$-module of all holomorphic functions on $D \times M$. $H(D \times M)$ is the $H^0(D \times M, \mathcal{O}_{D\times M})$-module of all m-column vector valued holomorphic functions on $D \times M$.

Let $Ker\,T$ and $Im\,T$ be, respectively, the kernel and image of the homomorphism

$$(1.7) \qquad T : H(D \times M) \longrightarrow H(D \times M).$$

As Theorem 3_m, we have proved the following theorem in the previous paper of the authors[9].

PREPARATION THEOREM. *The necessary and sufficient condition for $H(D \times M) = T(H(D \times M))$ is that, for $i = 1, 2, \ldots, m$, D_i is either a simply connected domain or a doubly connected domain in $\mathbb{C}$ with $H^0(D_i, Ker\,T_i) = 0$.*

In the present paper, we let each cross domain $D(r)$ varies as polycylinder valued function of the parameter $r \in M$ and instead of the product manifold, we consider a domain Ω in the product manifold $\mathbb{C}^m \times M$ of type

$$(1.8) \qquad \Omega := \{(z, r) \in \mathbb{C}^m \times M; z \in D(r), r \in M\}$$

where $D(r)$ is a polycylinder, product

$$(1.9) \qquad D(r) := D_1(r) \times D_2(r) \times \cdots \times D_m(r)$$

of open sets $D_1(r), D_2(r), \ldots$ and $D_m(r)$, not necessarily connected, in the complex plane $\mathbb{C}$. For each i with $1 \leq i \leq m$, let $a^i = (a^i_{jk}(z, r))$ be a square matrix of degree m whose each (j, k)-element $a^i_{jk}(z, r)$ is a holomorphic function in Ω. Let M be a Stein manifold and m be a positive integer. For any point r of M and each integer j with $1 \leq j \leq m$, let $D_j(r)$ be an open set, which may depends on r and which is not necessarily connected, in the complex plane $\mathbb{C}$. We put

$$(1.10) \qquad D(r) := D_1(r) \times D_2(r) \times \cdots \times D_m(r).$$

For each point r of M, each connected component of the opoen set $D(r)$ is a domain of holomorphy in the complex m-space $\mathbb{C}^m$. We assume that the subset

$$\Omega := \{(z,r) \in \mathbb{C}^m \times M; z \in D(r), r \in M\} \tag{1.11}$$

of the product Stein manifold $\mathbb{C}^m \times M$ forms a Stein domain in $\mathbb{C}^m \times M$. For each i with $1 \leq i \leq m$, we put

$$D^i(r) := D_1(r) \times D_2(r) \times \cdots \times D_{i-1}(r) \times D_{i+1}(r) \times \cdots \times D_m(r), \tag{1.12}$$

$$\Omega^i := \{(z', r) \in \mathbb{C}^{m-1} \times M; z' \in D^i(r), r \in M\}. \tag{1.13}$$

For each i with $1 \leq i \leq m$ and for any point $(z', r) := (z_1, z_2, \ldots, z_{i-1}, z_{i+1} \ldots, z_m, r)$ of Ω^i, we introduce in the set Ω an equivalence relation $\approx$ by regarding each connected component of $\{z_1\} \times \{z_2\} \times \cdots \times \{z_{i-1}\} \times D_i(r) \times \{z_{i+1}\} \times \ldots \{z_m\} \times \{r\}$ as a point. Let $\widetilde{\Omega^i} := \Omega / \approx$ be the factor space of Ω by this equivalence relation and $\varphi^i : \widetilde{\Omega^i} \to \Omega^i \subset \mathbb{C}^{m-1} \times M$ be the canonical mapping which is a local homeomorphism.

Let $a^i = (a^i_{jk}(z,r))$ be a square matrix of degree m whose each (j,k) element $a^i_{jk}(z,r)$ is a holomorphic function in Ω. For each i with $1 \leq i \leq m$, let T_i be a differential operator defined by (1.2).

Let $\mathcal{O}_\Omega$ be the sheaf of germs of all holomorphic functions on $\mathcal{O}_\Omega$. Then each differential operator T_i defines a sheaf homomorphism

$$T_i : (\mathcal{O}_\Omega)^m \to (\mathcal{O}_\Omega)^m \tag{1.14}$$

similarly as (1.4). We put

$$T = T_1 T_2 \cdots T_m. \tag{1.15}$$

Let

$$H(\Omega) = H^0(\Omega, (\mathcal{O}_\Omega)^m) \tag{1.16}$$

be the set of global sections of $(\mathcal{O}_\Omega)^m$ over Ω.

Under the above notations, the main results of the present paper is as follows:

MAIN THEOREM. *The necessary condition for* $H^1(\Omega, Ker\, T) = 0$ *is that, for any* $i = 1, 2, \ldots, m$, *each connected component of* $D_i(r)$ *is either, simultaneously for any* $r \in M$, *a simply connected domain in* $\mathbb{C}$ *or, simultaneously for any* $r \in M$, *a doubly connected domain in* $\mathbb{C}$ *for any* $r \in M$ *with* $H^0(D_i(r), Ker\, T) = 0$ *For* $i = 1, 2, \ldots, m$ *such that each connected component of* $D_i(r)$ *is simultaneously simply connected for any* $r \in M$, *we assume additionally that the coefficients* $a^i_{jk}(z, r)$*'s are holomorphic in a cylindrical neighborhood* E *of* Ω. *Then the necessary and sufficient condition for* $H^1(\Omega, Ker\, T) = 0$ *is that, for* $i = 1, 2, \ldots, m$, *the factor space* $\widetilde{\Omega^i}$ *of* Ω *is a Hausdorff space, that, for any* i *such that each connected component of* $D_i(r)$ *is simultaneously a simply connected domain in* $\mathbb{C}$ *for any* $r \in M$, *the domain* $(\widetilde{\Omega^i}, \varphi^i)$ *is a Stein domain over* $\mathbb{C}^{m-1} \times M$ *and that for any* i *such that each connected component of* $D_i(r)$ *is simultaneously a doubly connected domain in* $\mathbb{C}$ *for any* $(z', r) \in \mathbb{C}^{m-1} \times M$, *there holds* $H^0(\{z'\} \times D_i(r) \times M, Ker\, T_i) = 0$.

2. Proof of the main theorem

Let r be a point of the base space M and $g(z)$ be any element of $H^0(D(r), (\mathcal{O}_{D(r)})^m)$, that is, any vector-valued holomorphic function on the cylindrical open set $D(r)$ which is regarded as an analytic subset of the Stein manifold M, by the theorem of Oka-Cartan-Serre, there exists a holomorphic function $G(z, r)$ on the whole Ω such that restriction of $G(z, r)$ to the section $D(r)$ coincides with $g(z)$. The assumption that $H^1(\Omega, Ker\ T) = H(\Omega)/T(H(\Omega)) = 0$ implies the existence of a solution $F \in H(\Omega)$ of the partial differential equation $TF = G$. The restriction f of F to $D(r)$ is a solution of the partial differential equation $Tf = g$ because the partial differentiations of T is done only for the variables of z. Hence we have $H^1(D(r), Ker\ T) = H(D(r))/T(H(D(r))) = 0$. By the preparation theorem, that is, Main Theorem at page 181 of the previous paper [8], for any $r \in M$ and for any $i = 1, 2, \ldots, m$ each connected component of $D_i(r)$ is either a simply connected domain with $H^0(D_i(r), Ker\ T_i) = 0$ in $\mathbb{C}$ or a doubly connected domain in $\mathbb{C}$. Moreover, by the argument from page 102 till page 105 and the inductive argument from page 175 till page 177 of the previous paper [9], and for any $i = 1, 2, \ldots, m$ each connected component of $D_i(r)$ is simultaneously either a simply connected domain in $\mathbb{C}$ or simultaneously a doubly connected domain in $\mathbb{C}$ for all $r \in M$.

Under the assumption that $H^1(\Omega, Ker\ T) = 0$, for any i such that each connected component of $D_i(r)$ is simultaneously a simply connected domain in $\mathbb{C}$ for any $r \in M$, we put the additional assumption that the coefficients $a^i_{jk}(z,r)$'s are holomorphic in a cylindrical neighborhood E of Ω and we can prove by inductional method from page 174 till page 180 of the previos paper[9] that the first cohomology group of the topological space $\widetilde{\Omega^i}$ over $\mathbb{C}^{m-1} \times M$ with coefficients in the sheaf of germs of holomorphic function over $\widetilde{\Omega^i}$ vanishes and, therefore, we can prove by the arguments from page 108 till page 114 of the former paper[8] that $\widetilde{\Omega^i}$ is a Hausdorff space and, with respect to the complex structure induced canonically by the local homeomorphism φ^i, the complex manifold $\widetilde{\Omega^i}$ is Stein. Hence the domain $(\widetilde{\Omega^i}, \varphi^i)$ is a Stein domain over $\mathbb{C}^{m-1} \times M$.

Under the assumption that $H^1(\Omega, Ker\ T) = 0$, for any i such that each connected component of $D_i(r)$ is simultaneously a doubly connected domain in $\mathbb{C}$ for any $r \in M$, we can prove by the arguments from page 105 till 107 of the former paper[8] the T_2-ness of the topological space $\widetilde{\Omega^i}$.

We can also establish the validity of the converse part of the main theorem because, for any i such that each connected component of $D_i(r)$ is simultaneously a simply connected domain in $\mathbb{C}$, the condition given for the converse allows us to use the vanishment of the locally free analytic sheaf of sections of vector bundles formed by germs of homogeneous solutions f of $T_i f = 0$ and because, for any i such that each connected component of $D_i(r)$ is simultaneously a doubly connected domain in $\mathbb{C}$, the condition given for the converse $H^0(\{z'\} \times D_i(r) \times M, Ker\ T_i) = 0$ gives us the unity of the homogeneous solution f of $T_i f = 0$ in the cut $\{z'\} \times D_i(r) \times M$. Thus, we can prove $H^1(\Omega, Ker\ T_i) = H(\Omega)/T_i(H(\Omega)) = 0$ for any i and we can prove the validity of converse by induction with respect to i.

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