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A FREIHEITSSATZ FOR SEMIGROUPS

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1. Introduction

The Freiheitssatz for one-relator groups states that if G is the group defined by the presentation $\langle a_1, a_2, a_3, \cdots ; r\rangle$, where r is a cyclically reduced word in $\{a_1, a_2, a_3, \cdots\}$ containing a_1, then the subgroup of G generated by $\{a_2, a_3, \cdots\}$ is a free group. We can restate it as follow: if G is the group defined by the presentation $\langle X; r\rangle$, where r is a cyclically reduced word in X containing x_0 in X, then the free group generated by $X \setminus \{x_0\}$ is embedded in G by the natural homomorphism extending the map $x \mapsto x$ for all x in $X \setminus \{x_0\}$. This Freiheitssats for one-relatred groups is true ([7]).

One may want to generalize the Freiheitssatz to groups with more than one relators as follow: if G is the group defined by the presentation $\langle X; R\rangle$, where each r in R is a cyclically reduced word in X, and if X_0 is a subset of X and R_0 is the subset of R consisting of all elements which involves only the variables in X_0, then the group defined by the presentation $\langle X_0; R_0\rangle$ is embedded in G by the natural homomorphism. But this is not true in general as the following example shows.

Example 1. Let G be the group defined by the presentation $\langle x, y, z; xy^{-1}, yz\rangle$ and put $X_0 = \{x, z\}$. Then R_0 is empty and the group G_0 defined by the presentation $\langle X_0; R_0\rangle$ is the free group on X_0. However, the subgroup of G generated by $\{x, z\}$ is not a free group. In fact, in G, we have $xz = xy^{-1}yz = 1 \cdot 1 = 1$, and so $x = z^{-1}$. Thus the natural homomorphism is not an embedding.

In this paper, we want to state a Freiheitssatz for semigroups and provide a couple of conditions for this Freiheitssatz to hold.

By a *semigroup presentation* P, we mean a pair $[X; R]$ where X is a set of generators and R is a set of relations. Thus X consists of symbols $x_1, x_2, \cdots$ and R consists of pairs (u_i, v_i), $(i \in I)$, where u_i and v_i are

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semigroup words in X. It is customary to write $u = v$ instead of (u, v). We denote by $S(P)$ the semigroup defined the presentation $P = [X; R]$.

A FREIHEITSSATZ FOR SEMIGROUPS. *Let $P = [X; R]$ be a semigroup presentation. Suppose X_0 is a subset of X, R_0 the subset of R consisting of all relations involving only the variables in X_0. Then, for $P_0 = [X_0; R_0]$, $S(P_0)$ is embedded in $S(P)$ by the natural homomorphism, $x \mapsto x$ for all x in X_0.*

The main results of this paper are the following two theorems.

THEOREM 1. *Let $P = [X; u = v]$ be an one-relator semigroup presentation. Then the Freiheitssatz holds for $S(P)$. That is, if X_0 is the subset of X which does not contain a variable appearing in the relation $u = v$, then the subsemigroup of $S(P)$ generated by X_0 is free.*

THEOREM 2. *Let $P = [X; R]$ where the left and right sides of each relation $u = v$ in R involve the same variables. Then the Freiheitssatz holds for $S(P)$.*

We will prove Theorem 1 algebraically in §2 and introduce semigroup diagrams in §3, which will be used in §4 for a geometric proof of Theorem 2. At the end of the paper, we will show by an example that the condition in Theorem 2 can not be weaken.

2. Proof of Theorem 1

For a semigroup presentation $P = [X; u = v]$, let $\bar{P} = \langle X; uv^{-1} \rangle$ be the corresponding group presentation, and let $G(\bar{P})$ denote the group defined by $\bar{P}$. The following lemmas are trivial

LEMMA 1. *The map $S(P) \longrightarrow G(\bar{P})$ such that $x \mapsto x$ for all $x \in X$ can be extended to a homomorphism.*

LEMMA 2. *For any set X, the free semigroup on X can be embedded in the free group on X by the natural homomorphism such that $x \mapsto x$ for all $x \in X$.*

Now let $P = [X; u = v]$ be an one-relator semigroup presentation. Suppose X_0 is the subset of X which does not contain a variable appearing in the relation $u = v$. To prove Theorem 1, we need to show that subsemigroup of $S(P)$ generated by X_0 is free.

Suppose w_1 and w_2 are words in X_0 such that $w_1 = w_2$ in $S(P)$. It is enough to show that $w_1 = w_2$ in the free semigroup on X_0. By Lemma 1, $w_1 = w_2$ in $G(\bar{P})$. By Lemma 2 and the Freiheitssatz for the one-relator group $G(\bar{P})$, there is an embedding from the free semigroup on X_0 into $G(\bar{P})$. Thus $w_1 = w_2$ in the free semigroup on X_0. This completes the proof of Theorem 1.

3. Semigroup diagrams

By a *semigroup diagram* over a set X we mean a labelled oriented map M ([6], p.236]) with the following properties.

(1) M is connected and simply connected.

(2) Each edge of D is labelled with an element of X,

(3) There are two distinguished points on the boundary of M, which are denoted by $\sigma(M)$ and $\tau(M)$ respectively, and the boundary of M consists of two directed paths from $\sigma(M)$ to $\tau(M)$.

(4) There are two distinguished points on the boundary of each region Δ, which are denoted by $\sigma(\Delta)$ and $\tau(\Delta)$ respectively, and the boundary of Δ consists of two directed paths from $\sigma(\Delta)$ to $\tau(\Delta)$.

We can describe a semigroup diagrams M over X pictorially by drawing a figure. In doing so, we put $\sigma(M)$ at the far left and $\tau(M)$ at the far right of the figure. We also put each $\sigma(\Delta)$ to the left of $\tau(\Delta)$ (See figure 1).

By an obvious reason, the two directed paths from $\sigma(M)$ to $\tau(M)$ which constitute the boundary of M are called the *upper boundary* and the *lower boundary* of M, and denoted by $\mu(M)$ and $\lambda(M)$ respectively. Similarly, we define $\mu(\Delta)$ and $\lambda(\Delta)$ for region each Δ of M.

For each edge e, let $\varphi(e)$ denote the label of e, and extend φ over all directed paths of M, i.e., if $\alpha = e_1 e_2 \cdots e_n$ is a directed path then $\varphi(\alpha) = \varphi(e_1)\varphi(e_2)\cdots\varphi(e_n)$.

Given a semigroup prsentation $P = [X; R]$, a semigroup diagram M over X is called a *P-diagram* if the following holds.

(5) For each region Δ of M, either $\varphi(\mu(M)) = \varphi(\lambda(M))$ or $\varphi(\lambda(M)) = \varphi(\mu(M))$ is a relation in R.

For example, the semigroup diagram in Figure 1 is a P-diagram over the semigroup presentation

$$P = [x, y, z, t; xxy = yz,\ xzy = yz,\ yx = zy].$$

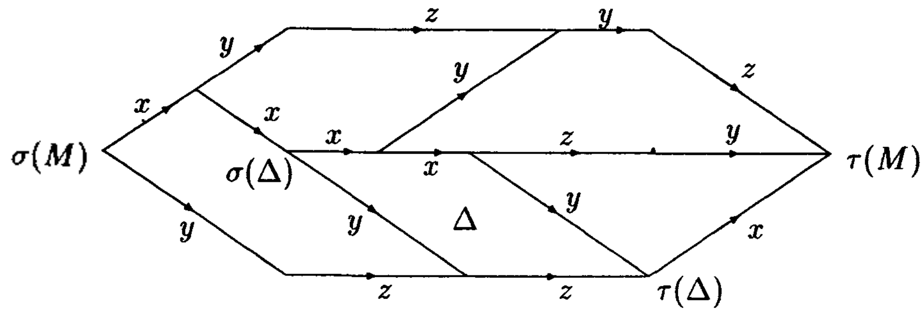

Figure 1

LEMMA 3 ([9]). *Let $P = [X; R]$ and w_1, w_2 be words in X. Then $w_1 = w_2$ in $S(P)$ if and only if there is a P-diagram M such that $\varphi(\mu(M)) = w_1$ and $\varphi(\lambda(M)) = w_2$.*

COROLLARY. *Let M be a P-diagram and v_1, v_2 be vertices of M. If α and β are directed paths from v_1 to v_2 then $\varphi(\alpha) = \varphi(\beta)$ in $S(P)$.*

For a semigroup diagram M, let $\#(M)$ denote the number of regions of M. If α and β are directed paths in M then write $\alpha \subseteq \beta$ to denote that α is a subpath of β.

LEMMA 4 ([4]). *Let M be a semigroup diagram and $\#(M) > 0$. Then there is a region Δ such that $\mu(\Delta) \subseteq \mu(M)$. Dually, there is a region Δ such that $\lambda(\Delta) \subseteq \lambda(M)$.*

Proof. [Another Proof]. Let Σ be the set of all regions of M. Define a relation '$>$' on Σ by $\Delta > \Phi$ if $\lambda(\Delta)$ and $\mu(\Phi)$ have a common edge, and let '$\geq$' be the reflexive and transitive closure of '$>$'. Then it can be seen that '$\geq$' is a partial ordering on Σ. Since Σ is finite, it has a maximal element Δ and a minimal element Φ. Then $\mu(\Delta) \subseteq \mu(M)$ and $\lambda(\Phi) \subseteq \lambda(M)$.

The geometric method using diagrams is well developed for combinatorial group theory and proved to be very useful ([2],[3], [6]). This method is also adapted by many people for the study of word problem and embedding problem of semigroups ([1],[5],[8],[9]).

4. Proof of Theorem 2

Let $P = [X; R]$ where R consists of relations whose left and right hand sides involve the same variables, and let $P_0 = [X_0, R_0]$ where X_0 is a subset of X and R_0 is a set of relations in R which involve only the variables in X_0. We need to show the natural homomorphism $S(P_0) \longrightarrow S(P)$ is injective. Suppose w_1 and w_2 are words in X_0 such that $w_1 = w_2$ in $S(P)$. We wish to show $w_1 = w_2$ in $S(P_0)$.

By Lemma 3 there is a P-diagram M such that $\varphi(\mu(M)) = w_1$ and $\varphi(\lambda(M)) = w_2$. We shall show that there is a P_0-diagram M^* such that $\varphi(\mu(M^*)) = w_1$ and $\varphi(\lambda(M^*)) = w_2$. We proceed by induction on $\#(M)$. If M has no region then the conclusion is trivial, because w_1 and w_2 are identical in this case (Figure 2(a)). If M has only one region then M itself is a P_0 diagram (Figure 2(b)).

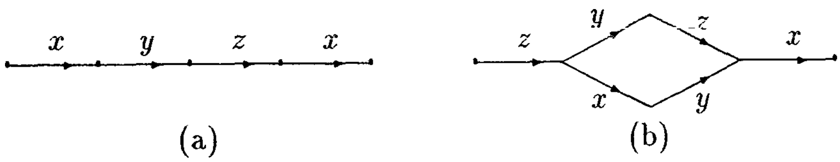

Figure 2

Assume the conclusion is true for P-diagrams with less than $\#(M)$ regions and we will show the conclusion is true for M. By Lemma 4 there is a region Δ such that $\mu(\Delta) \subseteq \mu(M)$ (Figure 3(a)).

Let M' be the P-diagram obtained from M by deleting $\mu(\Delta)$ and the interior of Δ. Note that the lower boundary $\lambda(\Delta)$ of Δ remains a part of the upper boundary $\mu(M')$ of M' (Figure 3(b)). Thus, if $\mu(M) = \alpha\mu(\Delta)\beta$ then $\mu(M') = \alpha\lambda(\Delta)\beta$. Here α and β may be empty. Note that $\varphi(\mu(M))$ is a word in X_0, and, since $\varphi(\mu(\Delta))$ and $\varphi(\lambda(\Delta))$ involve the same variables, $\varphi(\mu(M'))$ is a word in X_0. Let this word be w'. Since $\#(M') < \#(M)$, by induction hypothesis, there is a P_0 diagram M'' such that $\varphi(\mu(M'')) = w_1$ and $\varphi(\lambda(M'')) = w_2$ (Figure 3(c)). Now $\mu(M'') = \alpha'\beta'\gamma'$ for some directed paths α', β' and γ', where $\varphi(\beta') = \varphi(\lambda(\Delta))$. Let M^* be the semigroup diagram obtained by gluing Δ to M'' identifying $\lambda(\Delta)$ with β'. Then, M^* is a P_0-diagram such that $\varphi(\mu(M^*)) = w_1$ and $\varphi(\lambda(M^*)) = w_2$ (Figure 3(d)). This completes the proof.

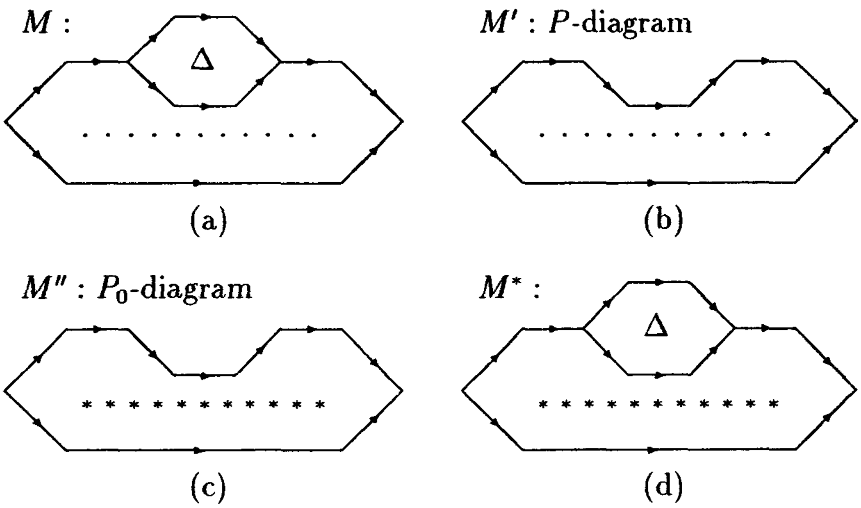

Figure 3

Remark. Without the condition that every relation in R involves the same variables on both sides, the Freiheitssatz does not holds, as the following example shows.

Example. Let $S(P)$ be defined by the presentation

$$P = [x, y, z, t; xy = tz, xz = zx, xt = tx]$$

and let $S(P_0)$ be the semigroup defined by $P_0 = [x, y, z; xz = zx]$. Then, $xyx = xxy$ in $S(P)$ since $xyx = tzx = txz = xtz = xxy$ (See Figure 4).

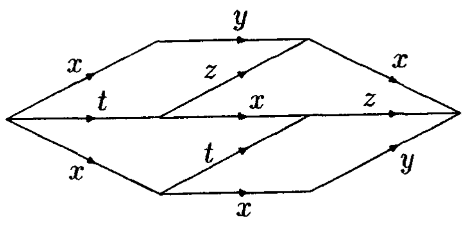

Figure 4

However, $xxy \neq xyx$ in $S(P_0)$, since the relation $xz = zx$ can be applied to neither xxy nor xyx. In fact,

$$S(P_0) = < y > * (< x > \oplus < z >).$$

Thus, $S(P_0)$ is not embeddable in $S(P)$ by the natural map.

References

1. Jung R. Cho, *E-unitary problem of certain inverse monoids*, submitted
2. Jung R. Cho & S.J. Pride, *Embedding semigroups into groups, and the asphericity of semigroups*, Inter. J of Alg. and Comp. **3** (1993), 1-13
3. D J. Collins & J Huebschmann, *Spherical diagrams and identities among relations*, Math. Ann **261** (1982), 155-183.
4. James Howie & Stephen J Pride, *A spelling theorem for staggered generalized 2-complexes*, Adv in Math. **36** (1980), 283-296
5. S V. Ivanov, *Relation modules and relation bimodules of groups, semigroups and associative algebras*, Inter J of Alg and Comp **1** (1991), 89-113
6. R C Lyndon & P E Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
7. W. Magunus, A Karras & D Solitar, *Combinatorial group theory, 2nd ed.*, Dover publ New York, 1976
8. S W. Margolis & J.C Meakin, *E-unitary inverse monoids and the Cayley graph of a group presentation*, J Pure and Appl Alg **58** (1989), 45-76
9. John H. Remmers, *On the Geometry of Semigroup Presentations*, Adv in Math. **36** (1980), 283-296

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