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THE LEAST POSITIVE EIGENVALUE OF LAPLACIAN FOR $SU(4)/SU(2)\otimes SU(2)$

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1. Introduction.

Let (M, g) be an n−dimensional compact Riemannian manifold. We denote by Δ the Laplace-Beltrami operator acting on the space $C^\infty(M)$ of all complex valued smooth functions on M, that is,

$$(1.1)\quad \Delta = -\sum_{i,j}\frac{1}{\sqrt{G}}\frac{\partial}{\partial x^i}(\sqrt{G}g^{ij}\frac{\partial}{\partial x^j}),$$

where the g_{ij} are the components of g with respect to a local coordinate $(x_1, x_2, \cdots, x_n)$, (g^{ij}) is the inverse matrix of (g_{ij}) and $G = \det(g_{ij})$. Then, the spectrum $Spec(M, g)$ of the Laplacian Δ, i.e, the set of all eigenvalues of the Laplacian, consists of

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \to +\infty.$$

The task that calculates the spectrum $Spec(M, g)$ seems to be impossible, in general, for nonhomogeneous Riemannian manifolds. For a few Riemannian manifolds, e.g., flat tori, lens spases and symmetric spaces, spectra have been calculated ([7],[8],[10]).

In this paper, we treat a normal homogeneous manifold $(M.g)$ $= SU(4)\,/\,SU(2)\otimes SU(2)$. That is, let $(\cdot,\cdot)$ be an $\mathrm{Ad}(SU(4))$−invariant inner product on the Lie algebra $\mathfrak{su}(4)$. Let $\mathfrak{m}$ be the orthogonal complement to the subalgebra $\mathfrak{su}(2)\otimes I_2 + I_2\otimes\mathfrak{su}(2)$ of $SU(2)\otimes SU(2)$ in $\mathfrak{su}(4)$ relative to $(\cdot,\cdot)$, so that $\mathfrak{su}(4) = \mathfrak{su}(2)\otimes I_2 + I_2\otimes\mathfrak{su}(2) + \mathfrak{m}$ and $\mathrm{Ad}(SU(2)\otimes SU(2))\,(\mathfrak{m}) = \mathfrak{m}$, where

$$a\otimes b := \begin{pmatrix} a_{11}b & a_{12}b \\ a_{21}b & a_{22}b \end{pmatrix},$$

$$(a = (a_{ij}), b = (b_{ij}) \in M_2(C)).$$

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The tangent space $T_o(SU(4)/SU(2)\otimes SU(2))$ at the origin $o{:=}SU(2)\otimes SU(2)$ can be identified with the subspace $\mathfrak{m}$ by

$$\mathfrak{m}\in X\to X_o\in T_o(SU(4)/SU(2)\otimes SU(2)),$$

where $X_o\ f:=d/dt\ f(\exp tX\cdot o)|_{t=0}$ for a C^∞-function f on $SU(4)/SU(2)\otimes SU(2)$. An inner product g_o on the tangent space at o defined by $g_o(X_o,Y_o)=(X,Y)$, $X,Y\in\mathfrak{m}$, can be uniquely extended to a $SU(4)$-invariant Riemannian metric g on $SU(4)/SU(2)\otimes SU(2)$.

2. The main result.

In this paper, we have

Theorem. *Let (M,g) be a normal homogeneous Riemannian manifold $(SU(4)/(SU(2)\otimes SU(2)),\ g)$ with the normal metric g which is canonically-induced from the Killing form B on the Lie algebra $\mathfrak{su}(4)$ of $SU(4)$. Then, the least positive eigenvalue of the Laplacian Δ_g for $(M,\ g)$ is $\frac{9}{8}$.*

3. Proof of the main result.

3.1. In this part, we present some results on the sectra for normal homogeneous Riemannian manifolds.

The spectrum $Spec(G/K,g)$ of the Laplacian for a normal homogeneous Riemannian manifold G/K can be obtained as follows [8, PP.979-980]. Let $\mathfrak{t}$ be a maximal abelian subalgebra of the Lie algebra $\mathfrak{g}$ of G. Since the weight of a finite unitary representation of G relative to $\mathfrak{t}$ has its value in purely imaginary numbers on $\mathfrak{t}$, we consider the weight as an element of $\sqrt{-1}\mathfrak{t}^*$, where $\mathfrak{t}^*$ denotes the real dual space of $\mathfrak{t}$. From the Ad(G)-invariant inner product on $\mathfrak{g}$, a positive inner product on $\sqrt{-1}\mathfrak{t}^*$ is defined in the usual way and denoted by the same symbol $(\cdot,\cdot)$. Fixing a lexicographic order $>$ on $\sqrt{-1}\mathfrak{t}^*$, let P be the set of all positive roots of the complexification $\mathfrak{g}^c$ of $\mathfrak{g}$ relative to $\mathfrak{t}^c$. We denote by δ half the sum of all elements in P. Let $\Gamma(G)=\{H\in\mathfrak{t};\exp H=e\}$ and $I=\{\lambda\in\sqrt{-1}\mathfrak{t}^*;\lambda(H)\in 2\sqrt{-1}Z$ for all $H\in\Gamma(G)\}$. An element in I is called a $G-integral\ form$. The elements of

$$D(G)=\{\lambda\in I;\ (\lambda,\alpha)\geq 0\text{ for all }\alpha\in P\}$$

are called *dominant* G-integral forms. Then there exists a natural bijection from $D(G)$ onto the set $\mathfrak{D}(G)$ of all nonequivalent finite dimensional irreducible unitary representation of G which map a dominant G-integral form $\lambda \in D(G)$ to an irreducible unitary representation (V_λ, π_λ) having highest weight λ. For $\lambda \in D(G)$, put $d(\lambda)$ the dimension of the representation V_λ. $d(\lambda)$ is given by

$$d(\lambda) = \Pi_{\alpha \in P} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)}.$$

A representation (V_λ, π_λ) in $\mathfrak{D}(G)$ is called *spherical* relative to K if there exists a nonzero vector $v \in V_\lambda$ such that $\pi_\lambda(k)\ v = v$ for all $k \in K$. Let $\mathfrak{D}(G,K)$ be the set of all spherical representations in $\mathfrak{D}(G)$ relative to K and $D(G,K) = \{\lambda \in D(G);\ (V_\lambda, \pi_\lambda) \in \mathfrak{D}(G,K)\}$.

Then the following Theorem is well known.

Theorem 1 [7, Propo. 2.1, P.558]. *The spectrum $Spec(G/K, g)$ of the Laplacian on the normal homogeneous space $(G/K, g)$ is given by eigenvalues* $(\lambda + 2\delta, \lambda)$, $\lambda \in D(G,K)$.

3.2. The inclusion of $SU(2) \otimes SU(2)$ into $SU(4)$ is the tensor product of two usual linear representations of $SU(2)$. In this section, we use the following notations.

$G = SU(4), \quad G_{(2)} = SU(2), \quad H = (SU(2) \otimes SU(2)), \quad M = G/H,$

$T = \{diag[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4]; \epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1, |\epsilon_i| = 1, \epsilon_i \in C\},$

$T_{(2)} = \{diag[\epsilon_1, \epsilon_2]; \epsilon_1\epsilon_2 = 1, |\epsilon_i| = 1, \epsilon_i \in C\},$

$\mathfrak{g}$ (resp. $\mathfrak{g}_{(2)}$) : the Lie algebra of G (resp. $G_{(2)}$),

$\mathfrak{h} = \mathfrak{su}(2) \otimes I_2 + I_2 \otimes \mathfrak{su}(2)$: the Lie algebra of H as a subspace of $\mathfrak{g}$,

$\mathfrak{t}$ (resp. $\mathfrak{t}_{(2)}$) : the Lie algebra of T (resp. $T_{(2)}$),

$\mathfrak{g}^c$ (resp. $\mathfrak{t}^c$) : the complexification of $\mathfrak{g}$ (resp. $\mathfrak{t}$),

$diag[\epsilon_1, \epsilon_2, \cdots, \epsilon_n]$: a diagonal matrix with diagonal elements $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$.

We give an Ad(G)–invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ by

$$(3.1)\quad (X,Y) = -B(X,Y) = -8\ Trace(XY), \quad (X, Y \in \mathfrak{g}),$$

where B is the Killing form on $\mathfrak{g}^c$. Let g be the G–invariant Riemannian metric on M induced from this inner product $(\cdot,\cdot)$. We denote by $e_j \in \sqrt{-1}\mathfrak{t}^*$ (j=1,2,3,4), the Linear map

$$\sqrt{-1}\mathfrak{t} \ni diag[x_1, x_2, x_3, x_4] \longrightarrow x_j \in C.$$

Put $\alpha_i = e_i - e_{i+1}$, $(i = 1,2)$. We fix an lexicographic order $<$ on $\sqrt{-1}\mathfrak{t}^*$ in such a way $e_1 > e_2 > e_3 > 0 > e_4$. The set $D(G)$ of all dominant G-integral forms is given by

$$D(G) = \{\lambda = \sum_{i=1}^{3} m_i e_i;\ m_1 \geq m_2 \geq m_3 \geq 0,\ \text{each } m_j \in Z\ \}.$$

On the other hand, the elements $H_{e_j} \in \sqrt{-1}\mathfrak{t}$ such that $e_j(H) = B(H_{e_j}, H)$ for all $H \in \mathfrak{t}^c$ are given as follows:

$$(3.2)\quad \begin{cases} H_{e_1} = 1/32\ diag[3,-1,-1,-1],\ H_{e_2} = 1/32\ diag[-1,3,-1,-1], \\ H_{e_3} = 1/32\ diag[-1,-1,3,-1],\ H_{e_4} = 1/32\ diag[-1,-1,-1,3], \\ H_{\alpha_1} = 1/8\ diag[1,-1,0,0], \quad H_{\alpha_2} = 1/8\ diag[0,1,-1,0], \\ H_{\alpha_3} = 1/8\ diag[0,0,1,-1]. \end{cases}$$

Then the inner product $(\cdot,\cdot)$ induced on $\sqrt{-1}\mathfrak{t}$ is given by

$$(3.3)\quad (e_i, e_j) = (H_{e_i}, H_{e_j}) = \begin{cases} \frac{3}{32} & (i = j), \\ \frac{-1}{32} & (i \neq j), \end{cases}$$

where $i, j = 1,2,3,4$. The set P of all positive roots of $\mathfrak{g}^c$ relative to $\mathfrak{t}^c$ is

$$(3.4)\quad P = \{e_i - e_j; 1 \leq i < j \leq 4\},$$

so we have

$$(3.5)\quad \delta = 3e_1 + 2e_2 + e_3.$$

Therefore we have

$$(3.6)\quad \begin{aligned}(\lambda + 2\delta, \lambda) = (1/32)[&(m_1 - m_2)^2 + (m_2 - m_3)^2 + (m_3 - m_1)^2 \\ &+ {m_1}^2 + {m_2}^2 + {m_3}^2 + 12m_1 + 4(m_2 - m_3)]\end{aligned}$$

for $\lambda = m_1e_1 + m_2e_2 + m_3e_3 \in D(G)$. Moreover, we have

$$
(3.7) \quad \begin{aligned} d(\lambda) &= \Pi_{1\le i<j\le 4}\frac{(e_i - e_j, \lambda+\delta)}{(e_i - e_j, \delta)} \\ &= (1/12)(m_1+3)(m_2+2)(m_3+1) \\ &\qquad (m_1 - m_2 + 1)(m_2 - m_3 + 1)(m_1 - m_3 + 2) \end{aligned}
$$

for $\lambda = m_1 e_1 + m_2 e_2 + m_3 e_3 \in D(G)$. Here we have

Lemma 2. *Let* $\mathfrak{m}$ *be the orthogonal complement of* $\mathfrak{h}$ *in* $\mathfrak{g}$ *with respect to the inner product* $(\cdot,\cdot)$. *Then* $\mathfrak{m}$ *is given by*

(3.8) $\mathfrak{m} = \{(A_{ij}) \in \mathfrak{g};\ Trace A_{ij} = 0\ (i,j = 1,2),\ A_{11} + A_{22} = O_2\}$,

where O_2 *is the zero matrix of order* 2.

Proof. Since $\mathfrak{h} = \{X \otimes I_2 + I_2 \otimes Y;\ X, Y \in \mathfrak{g}_{(2)}\}$, $\mathfrak{m}$ is perpendicular to $\mathfrak{h}$. Moreover, $dim\mathfrak{h} + dim\mathfrak{m} = dim\mathfrak{g}$. Hence, the proof of this Lemma is completed.

In the unitary irreducible representations of $G_{(2)}$, we use the same symbols as occured in the unitary irreducible representation of G. Let ${V^{(2)}}_l$ be a unitary irreducible representation space of $G_{(2)}$ with highest weight le_1, where $le_1 \in D(G(2)) = \{me_1;\ m \geq 0, m \in Z\}$, [5, Th.1, P.46]. By the character formula of Weyl [10, PP.332-333] for $\lambda = f_1 e_1 + f_2 e_2 + f_3 e_3 \in D(G)$,

(3.9) $\chi_\lambda(h) = |\epsilon_i^{l_j}|/\xi(h)$

for each $h = diag[\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4] \in T$, where $|\epsilon_i^{l_j}|$ is the determinant of matrix of order 4 whose $(i,j)-$entries are $\epsilon_i^{l_j}$,

(3.10) $l_j = f_j + 4 - j \quad (j = 1,2,3)$, and $l_4 = 0$,

and $\xi(h)$ is given as follows:

(3.11) $\xi(h) = \Pi_{1\le i<j\le 4}(\epsilon_i - \epsilon_j)$.

Now let us consider the decomposition of V_λ, $(\lambda = \sum_{i=1}^4 f_i e_i \in D(G))$, into $H-$irreducible submodule as follows:

(3.12) $V_\lambda = \sum N(\lambda, l_1, l_2)\, {V^{(2)}}_{l_1} \otimes {V^{(2)}}_{l_2}$, where l_1, l_2 run over the set of all non-zero integers, $V^{(2)}{}_{l_1}\otimes V^{(2)}{}_{l_2}$ are irreducible representation spaces of $G_{(2)}\otimes G_{(2)}$, and $N(\lambda, l_1, l_2)$ is the multiplicity of $V^{(2)}{}_{l_1}\otimes V^{(2)}{}_{l_2}$ in V_λ.

We investigate $\lambda\in D(G)$ which belong to $D(G,H)$. $\lambda(\in D(G))$ belongs to $D(G,H)$ if and only if the unitary irreducible representation space V_λ of G contains $V^{(2)}{}_0\otimes V^{(2)}{}_0$. We put

$$\begin{aligned} h = h_1\otimes h_2 &= diag[x, x^{-1}]\otimes diag[y, y^{-1}] \\ &= diag[xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}] \\ &\in T_{(2)}\otimes T_{(2)}\subset T, \end{aligned}$$

then we have from (3,12)

$$(3.13)\quad \chi_\lambda(h) = \sum N(\lambda, l_1, l_2)\ \chi_{l_1}(h_1)\ \chi_{l_2}(h_2),$$

where χ_λ (resp. χ_{l_i}) is the character of the irreducible representation of G (resp. $G_{(2)}$) with the highest weight λ (resp. l_ie_1). Then we have

Lemma 3.

(a) $V_{e_1} = V^{(2)}{}_1\otimes V^{(2)}{}_1$,

(b) $V_{e_1+e_2} = V^{(2)}{}_2\otimes V(2)_0 + V^{(2)}{}_0\otimes V^{(2)}{}_2$,

(c) $V_{e_1+e_2+e_3} = V^{(2)}{}_1\otimes V^{(2)}{}_1$,

(d) $V_{2e_1} = V^{(2)}{}_2\otimes V^{(2)}{}_2 + V^{(2)}{}_0\otimes V^{(2)}{}_0$,

(e) $V_{2e_1+e_2} = V^{(2)}{}_3\otimes V^{(2)}{}_1 + V^{(2)}{}_1\otimes V^{(2)}{}_3 + V^{(2)}{}_1\otimes V^{(2)}{}_1$,

(f) $V_{2e_1+e_2+e_3} = V^{(2)}{}_2\otimes V^{(2)}{}_2 + V^{(2)}{}_2\otimes V^{(2)}{}_0 + V^{(2)}{}_0\otimes V^{(2)}{}_2$,

(g) $V_{2e_1+2e_2+e_3} = V^{(2)}{}_3\otimes V^{(2)}{}_1 + V^{(2)}{}_1\otimes V^{(2)}{}_3 + V^{(2)}{}_1\otimes V^{(2)}{}_1$,

(h) $V_{2e_1+2e_2+2e_3} = V^{(2)}{}_2\otimes V^{(2)}{}_2 + V^{(2)}{}_0\otimes V^{(2)}{}_0$.

Proof. Comparing with coefficients of both sides of (3.13) by using Weyl's character formular (3.9)-(3.11), we can obtain this Lemma. Q.E.D.

Remark. Comparing with the dimensions of both sides in the decompositions in the above Lemma, we can check these decompositions.

Using (3.6), we get

Lemma 4.

(a) $(2\delta + 2e_1, 2e_1) = 4(\delta + e_1 + e_2 + e_3, e_1 + e_2 + e_3) = 9/8$,

(b) In case of $\lambda \in \{m_1e_1 + m_2e_2 + m_3e_3 \in D(G);\ m_1 \geq 3\}$,
$(2\delta + \lambda, \lambda) > (39/32)$.

Therefore, we get from Theorem 1, Lemma 3 and Lemma 4 that the least positive eigenvalue of the Laplace-Beltrami operator Δ_g of $(G/H, g)$ is 9/8.

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