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SIEGEL MODULAR FORMS AND THETA SERIES (*)

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1. Introduction

Let $\mathcal{H}_g$ be the Siegel upper half plane of degree g(g is positive integer) and let $\Gamma_g = Sp(g, \mathbb{Z})$ be the Siegel modular group of degree g. If n is positive integer, then $\Gamma_g^{(n)} = \{M \in \Gamma_g \mid M \equiv E_{2g} \bmod n\}$ is called the *principal congruence subgroup* of degree g and of level n. The *Igusa group* is $\Gamma_g^{(n,2n)}=\{M \in \Gamma_g^{(n)} \mid$ the diagonals of $A{}^tB/n$ and $C{}^tD/n$ are even $\}$. Obviously $\Gamma_g = \Gamma_g^{(1)}$, and the *theta group* Γ_g^{θ} is Igusa group $\Gamma_g^{(1,2)}$. In this work we prove the generalized forms of theta series are modular forms and the mixed theta series, the basis of the vector space of all auxiliary theta series, are modular forms. We denote $F^{(k,j)}$ the set of all $k \times j$ matrices with entries in the commutative ring F. $\sigma(M)$ denotes the trace of a matrix M and ${}^tM$ denotes the transpose of M.

2. Siegel modular forms

Definition 2.1. Let $\Gamma \subset Sp(g, \mathbb{Z})$ be a subgroup of finite index. Then a *modular form of weight $k \in \mathbb{Z}$ and level* Γ is a holomorphic function f on Siegel's upper half space $\mathcal{H}_g$ to $\mathbb{C}$ such that for all $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$ we have

$$f(M < \Omega >) = \det(C\Omega + D)^k f(\Omega), \tag{1}$$

where $M < \Omega >= (A\Omega + B)(C\Omega + D)^{-1}$. And f is bounded in any domain $Y \geq Y_0 > 0$ in the case $g = 1$. In the case $g > 1$, f is bounded by the Koecher's priciple. The set of modular forms of weight k and level Γ is a vector space and is denoted by $[\Gamma, k]$. The product of two modular forms is a modular form, i.e.,

$$f \in [\Gamma, r],\ \ g \in [\Gamma, s] \Rightarrow f \cdot g \in [\Gamma, r + s].$$

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Siegel's Φ−operator defines a linear map $\Phi : [\Gamma_g, k] \longrightarrow [\Gamma_{g-1}, k]$ by

$$(\Phi f)(\Omega_1) = \lim_{t \to \infty} f \begin{pmatrix} \Omega_1 & 0 \\ 0 & it \end{pmatrix}, \quad \Omega_1 \in \mathcal{H}_{g-1}, \quad f \in [\Gamma_g, k].$$

A modular form $f \in [\Gamma_g, k]$ is called *cusp form* if it is in the kernel of Siegel's Φ−operator.

The basic problem is to find a finite system of theta series generating the space $[\Gamma, \frac{k}{2}]$.Böcherer([B]), Weissauer([W]) answered positively for the space $[\Gamma_g^{(1,2)}, \frac{k}{2}]$ with $k \geq 4g$, and $[\Gamma_g^{(q,2q)}, \frac{k}{2}]$ for small $k \leq \frac{g}{2}$ with arbitrary q. But the case $k < g$ we don't have a concrete answer.

If we choose

$$M = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix} \in \Gamma, \quad \text{where } S = {}^tS \text{ is integral},$$

then $f(\Omega + S) = f(\Omega)$, i.e., f is periodic, and therefore $f(\Omega)$ has a Fourier expansion of the form

$$f(\Omega) = \sum_{\substack{{}^tT = T \geq 0 \\ \text{half integral}}} c(T)\, e^{2\pi i \sigma(T\Omega)}, \quad \Omega \in \mathcal{H}_g. \tag{2}$$

If $c(T) \neq 0$ implies $T > 0$, we call $f \in [\Gamma_g, k]$ a cusp form, and $c(T) \neq 0$ implies $det\, T = 0$, then we call f a *singular modular form.* Resnikoff([R]) proved that $f \not\equiv 0$, $\Gamma \subset \Gamma_g$, f is singular if and only if $k < g$. We could define Siegel modular forms with character by inserting a factor ζ_M^{-1} in (1) in the usual way.

Lemma 2.2. *The group $Sp(g, \mathbb{R})$ acts on the space $\mathcal{H}_g \times \mathbb{C}^{(h,g)}$ by the maps :*

$$(\Omega, Z) \longmapsto (M < \Omega >, Z(C\Omega + D)^{-1}) \tag{3}$$

And the group $Sp(g, \mathbb{Z})$ acts on $\mathbb{Z}^{2g}$ by

$$(n_1, n_2) \longmapsto (Dn_1 - Cn_2, -Bn_1 + An_2),$$

and also on $\mathcal{H}_g \times \mathbb{C}^{(h,g)}$ by the maps (3).

The proof is easy.

3. Theta series

Definition 3.1. We define the theta series of index $\mathcal{M}$($\mathcal{M} \in \mathbb{R}^{(h,h)}, \mathcal{M} = {}^t\mathcal{M}$) with characteristic $(A,B) \in \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)}$ by

(4)

$$\begin{aligned}\vartheta^{\mathcal{M}}_{A,B}(\Omega, Z) &= \vartheta^{\mathcal{M}}\begin{bmatrix} A \\ B \end{bmatrix}(\Omega, Z) \\ &= \sum_{N \in \mathbb{Z}^{(h,g)}} \exp\{\pi i \sigma(\mathcal{M}(N+A)\Omega{}^t(N+A) + 2(N+A)^t \\ &\qquad (Z+B)))\}\end{aligned}$$

for $\Omega \in \mathcal{H}_g,\ Z \in \mathbb{C}^{(h,g)}$.

The series (4) converges absolutely and uniformly on any compact subset of $\mathcal{H}_g \times \mathbb{C}^{(h,g)}$. In particular we have the transformation formula for theta series

$$\vartheta^{\mathcal{M}^{-1}}_{A,B}(\Omega, 0) = e^{-\pi i \sigma({}^tAB)}(det\mathcal{M})^{\frac{g}{2}}(det(-i\Omega))^{\frac{h}{2}}\vartheta^{\mathcal{M}}_{B,-A}(\Omega, 0). \tag{5}$$

As in the vector-variable theory, the fundamental fact is a functional equation for ϑ for the action of $Sp(g,\mathbb{Z})$ on both variables $\Omega \in \mathcal{H}_g$ and $Z \in \mathbb{C}^{(h,g)}$. We now prove that the theta series $\vartheta^{\mathcal{M}}(\Omega, Z) = \vartheta^{\mathcal{M}}_{0,0}(\Omega, Z) = \vartheta^{\mathcal{M}}\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\Omega, Z)$ satisfies the following, i.e. a modular form of half weight:

$$\begin{aligned}&\vartheta^{\mathcal{M}}_{0,0}(M < \Omega >, Z(C\Omega + D)^{-1}) \\ &= \zeta_M \det(C\Omega + D)^{\frac{1}{2}} \cdot e^{\{\pi i \sigma(\mathcal{M}Z(C\Omega+D)^{-1}C{}^tZ\}} \cdot \vartheta^{\mathcal{M}}_{0,0}(\Omega, Z),\end{aligned} \tag{6}$$

where $\zeta_M^8 = 1$, and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(g,\mathbb{Z})$ satisfies

(i) diagonal $({}^tAC)$ even, diagonal $({}^tBD)$ even,

(ii) $\mathcal{M} \in \mathbb{Z}^{(h,h)},\ {}^t\mathcal{M} = \mathcal{M}$, and positive definite.

More generally let $\mathcal{M}$ be a symmetric, positive definite, and integral matrix of degree h and let $\mathcal{N}$ be a complete system of representatives of the cosets $(\mathcal{M})^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$. For $a \in \mathcal{N}$, we have

$$\begin{aligned}\vartheta^{\mathcal{M}}_{a,0}(-\Omega^{-1}, Z\Omega^{-1}) &= \{det(\mathcal{M})\}^{-\frac{g}{2}}\{det(-i\Omega)\}^{\frac{h}{2}} e^{\pi i \sigma(\mathcal{M}Z\Omega^{-1}{}^tZ)} \\ &\quad \sum_{b \in \mathcal{N}} e^{-2\pi i \sigma(\mathcal{M}b{}^ta)}\vartheta^{\mathcal{M}}_{b,0}(\Omega, Z).\end{aligned} \tag{6'}$$

To prove (6) for all $Z \in \mathbb{C}^{(h,g)}$, it certainly suffices to do so for $Z = X\Omega + Y$, $X, Y \in \mathbb{Q}^{(h,g)}$ by the density. So as a first step, we substitute $X\Omega + Y$ for Z and rewrite (6) for $\vartheta^{\mathcal{M}}\begin{bmatrix} X \\ Y \end{bmatrix}(\Omega, 0)$. We show that (6) is equivalent to :

Lemma 3.2.

(7)

$$\vartheta^{\mathcal{M}}\begin{bmatrix} X{}^tD - Y{}^tC \\ -X{}^tB + Y{}^tA \end{bmatrix}(M < \Omega >, 0)$$
$$= \zeta_M \det(C\Omega + D)^{\frac{1}{2}} \exp\{-\pi i\sigma(\mathcal{M}(X{}^tBD{}^tX - 2X{}^tBC{}^tY$$
$$+ Y{}^tAC{}^tY))\} \cdot \vartheta^{\mathcal{M}}\begin{bmatrix} X \\ Y \end{bmatrix}(\Omega, 0).$$

(ζ_M *depending only on* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *not on* X, Y, Ω)

Proof. Since $(C\Omega + D)^{-1} = -{}^tCM < \Omega > + {}^tA$ and

$$(X\Omega + Y)(C\Omega + D)^{-1} = (X{}^tD - Y{}^tC)M < \Omega > +(-X{}^tB + Y{}^tA),$$

we have by lemma 2.2,

$$\vartheta^{\mathcal{M}}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\Omega, 0) = \exp\{\pi i\sigma(\mathcal{M}(\alpha\Omega{}^t\alpha + 2\alpha{}^t\beta))\}\vartheta^{\mathcal{M}}(\Omega, \alpha\Omega + \beta),$$

that is,

$$\vartheta^{\mathcal{M}}(\Omega, \alpha\Omega + \beta) = \exp\{-\pi i\sigma(\mathcal{M}(\alpha\Omega{}^t\alpha + 2\alpha{}^t\beta))\}\vartheta^{\mathcal{M}}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}(\Omega, 0).$$

From this formula,

(8)

$$\vartheta^{\mathcal{M}}(M < \Omega >, (X\Omega + Y)(C\Omega + D)^{-1})$$
$$= \vartheta^{\mathcal{M}}(M < \Omega >, (X{}^tD - Y{}^tC)M < \Omega > +(-X{}^tB + Y{}^tA))$$
$$= \exp\{-\pi i\sigma(\mathcal{M}(X{}^tD - Y{}^tC)M < \Omega > {}^t(X{}^tD - Y{}^tC) + 2\mathcal{M}(X{}^tD$$
$$- Y{}^tC){}^t(-X{}^tB + Y{}^tA))\} \cdot \vartheta^{\mathcal{M}}\begin{bmatrix} X{}^tD - Y{}^tC \\ -X{}^tB + Y{}^tA \end{bmatrix}(M < \Omega >, 0).$$

On the other hand from the formula (7)

(9)

$$
\begin{aligned}
&\vartheta^{\mathcal{M}}(M<\Omega>,(X\Omega+Y)((C\Omega+D)^{-1})\\
&=\zeta_M \det(C\Omega+D)^{\frac{1}{2}}\exp\{\pi\imath\sigma(\mathcal{M}(X\Omega+Y)(C\Omega+D)^{-1}C{}^t(X\Omega\\
&\quad+Y))\}\cdot\vartheta^{\mathcal{M}}(\Omega,X\Omega+Y)\\
&=\zeta_M \det(C\Omega+D)^{\frac{1}{2}}\exp\{\pi i\sigma(\mathcal{M}(X\Omega+Y)(C\Omega+D)^{-1}C{}^t(X\Omega\\
&\quad+Y))\}\cdot\exp\{-\pi i\sigma(\mathcal{M}(X\Omega{}^tX+2X{}^tY))\}\vartheta^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega,0).
\end{aligned}
$$

Hence from the extreme right hand side of the formulas (8) and (9), we obtain

$$
\begin{aligned}
&\vartheta^{\mathcal{M}}\begin{bmatrix}X{}^tD-Y{}^tC\\-X{}^tB+Y{}^tA\end{bmatrix}(M<\Omega>,0)\\
&=\zeta_M\det(C\Omega+D)^{\frac{1}{2}}\exp\{\pi\imath\sigma(\mathcal{M}((X\Omega+Y)(C\Omega+D)^{-1}C{}^t(X\Omega+Y)\\
&\quad-(X\Omega{}^tX+2X{}^tY)+(X{}^tD-Y{}^tC)M<\Omega>{}^t(X{}^tD-Y{}^tC)\\
&\quad+2(X{}^tD-Y{}^tC){}^t(-X{}^tB+Y{}^tA)))\}\,\vartheta^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega,0)\\
&=\zeta_M\det(C\Omega+D)^{\frac{1}{2}}\exp\{-\pi\imath\sigma(\mathcal{M}(X{}^tBD{}^tX-2X{}^tBC{}^tY\\
&\quad+Y{}^tAC{}^tY))\}\vartheta^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega,0),
\end{aligned}
$$

using the basic facts on $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in Sp(g,\mathbb{Z})$, that is,

$$
{}^tDA-{}^tBC=E_g,\ \ {}^tDB={}^tBD,\ \ {}^tCA={}^tAC.
$$

However, the above calculation would have come out even simpler if, instead of

$$
\vartheta^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega,0)=\exp\{\pi i\sigma(\mathcal{M}(X\Omega{}^tX+2X{}^tY))\}\vartheta^{\mathcal{M}}(\Omega,X\Omega+Y),
$$

we use a modified ϑ that we will denote by $\tilde{\vartheta}$:

$$
(10)\quad \tilde{\vartheta}^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega)=\exp\{\pi i\sigma(\mathcal{M}(X\Omega{}^tX+X{}^tY))\}\vartheta^{\mathcal{M}}(\Omega,X\Omega+Y),
$$

written out

(10')

$$\tilde{\vartheta}^{\mathcal{M}}\begin{bmatrix} X \\ Y \end{bmatrix}(\Omega) = \sum_{N\in\mathbb{Z}^{(h,g)}} \exp\{\pi\imath\sigma(\mathcal{M}(N+X)\Omega{}^t(N+X) + 2\mathcal{M}(N+\frac{X}{2}){}^tY\}$$

If we use $\tilde{\vartheta}$ instead of ϑ the messy exponential factor is rather :

$$\pi\imath\sigma\{\mathcal{M}(X\Omega+Y)(C\Omega+D)^{-1}C{}^t(X\Omega+Y) - X\Omega{}^tX - X{}^tY + (X{}^tD - Y{}^tC)M<\Omega> {}^t(X{}^tD-Y{}^tC) + (X{}^tD - Y{}^tC)(-X{}^tB + Y{}^tA))\},$$

which turns out to vanish identically. Thus the functional equation (7) becomes :

(11)

$$\tilde{\vartheta}^{\mathcal{M}}\begin{bmatrix} X{}^tD - Y{}^tC \\ -X{}^tB + Y{}^tA \end{bmatrix}(M<\Omega>,0) = \zeta_M \det(C\Omega+D)^{\frac{1}{2}}\,\tilde{\vartheta}^{\mathcal{M}}\begin{bmatrix} X \\ Y \end{bmatrix}(\Omega)$$

Lemma 3.3. *$Sp(g,\mathbb{Z})$ acts on $\mathbb{Z}^{(h,g)} \times \mathbb{Z}^{(h,g)}$ as follow:*

$$(X,Y) \longmapsto (X{}^tD - Y{}^tC, -X{}^tB + Y{}^tA)$$

Proof. Let M, M' be in $Sp(g,\mathbb{Z})$ such that

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad M' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \quad \tilde{M} = MM' = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}.$$

Then

$$(X,Y) \longmapsto ((X{}^tD' - Y{}^tC'){}^tD - (-X{}^tB' + Y{}^tA'){}^tC, -(X{}^tD' - Y{}^tC'){}^tB + (-X{}^tB' + Y{}^tA'){}^tA) = (X{}^t\tilde{D} - Y{}^t\tilde{C}, -X{}^t\tilde{B} + Y{}^t\tilde{A})$$

Hence from lemmas 2.2 and 3.3 we conclude that $Sp(g,\mathbb{Z})$ acts as follows :

(i) On $\mathbb{Z}^{(h,g)}\times\mathbb{Z}^{(h,g)}$, by $(X,Y) \longmapsto (X{}^tD - Y{}^tC, -X{}^tB + Y{}^tA)$,

(ii) On $\mathcal{H}_g$, by $\Omega \longmapsto (A\Omega+B)(C\Omega+D)^{-1} = M<\Omega>$,

(iii) On $\mathbb{C}^{(h,g)}$, by $Z \longmapsto Z(C\Omega+D)^{-1}$.

Thus the functional equation for ϑ asserts that, up to an 8th root of 1, $\tilde{\vartheta}^{\mathcal{M}}_{X,Y}(\Omega)\sqrt{dZ_{11}\wedge\cdots\wedge dZ_{hg}}$ is invariant under $\Gamma_g^{(1,2)}\subset Sp(g,\mathbb{Z})$, that is,

(12)

$$\tilde{\vartheta}^{\mathcal{M}}[X{}^tD-Y{}^tC-X{}^tB+Y{}^tA](M<\Omega>)\sqrt{dW_{11}\wedge\cdots\wedge dW_{hg}}$$

$$=\zeta_M\tilde{\vartheta}^{\mathcal{M}}\begin{bmatrix}X\\Y\end{bmatrix}(\Omega)\sqrt{dZ_{11}\wedge\cdots\wedge dZ_{hg}},$$

where $W=Z(C\Omega+D)^{-1}$, $W,Z\in\mathbb{C}^{(h,g)}$, and

$$\Gamma_g^{(n)}=\{M\in Sp(g,\mathbb{Z})\mid M\equiv E_{2g}\mod n\},$$

$$\Gamma_g^{(2)}\subset\Gamma_g^{(1,2)}\subset\Gamma_g^{(1)}=Sp(g,\mathbb{Z}),$$

and the factor $\det(C\Omega+D)^{\frac{1}{2}}$ has been eliminated.

Since $\Gamma_g^{(1,2)}$ is generated by

$$\begin{pmatrix}0&-E_g\\E_g&0\end{pmatrix},\quad\begin{pmatrix}A&0\\0&{}^tA^{-1}\end{pmatrix},\quad\begin{pmatrix}E_g&B\\0&E_g\end{pmatrix}$$

for all $A\in GL(g,\mathbb{Z})$, symmetric integral B with even diagonal ([M]).

Now we consider the two cases to prove the functional equation (6) since the symplectic group $Sp(g,\mathbb{R})$ is generated by

$$\begin{pmatrix}0&-E_g\\E_g&0\end{pmatrix},\qquad\begin{pmatrix}E_g&B\\0&E_g\end{pmatrix},B={}^tB.$$

Case I. $M=\begin{pmatrix}E_g&B\\0&E_g\end{pmatrix}$, where B is symmetric matrix. Then the formula (6) reduces to :

$$\vartheta^{\mathcal{M}}(\Omega+B,Z)=\zeta_M\cdot\vartheta^{\mathcal{M}}(\Omega,Z)$$

Here we may take $\zeta_M=1$, because

$$\begin{aligned}\vartheta^{\mathcal{M}}(\Omega+B,Z)&=\sum_{N\in\mathbb{Z}^{(h,g)}}\exp\{\pi i\sigma(\mathcal{M}(N(\Omega+B){}^tN+2Z{}^tN))\}\\&=\sum_N\exp\{\pi i\sigma(\mathcal{M}NB{}^tN)\}\exp\{\pi i\sigma(\mathcal{M}(N\Omega{}^tN\\&\qquad+2Z{}^tN))\}\\&=\vartheta^{\mathcal{M}}(\Omega,Z),\end{aligned}$$

and $\sigma(\mathcal{M}NB{}^tN)$ is a character.

Case II. $M = \begin{pmatrix} 0 & -E_g \\ E_g & 0 \end{pmatrix}$. The formula (6) reduces to :

(13)
$$\vartheta^{\mathcal{M}}(-\Omega^{-1}, Z\Omega^{-1}) = \zeta_M \det(\Omega)^{\frac{1}{2}} \cdot \exp\{\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\vartheta^{\mathcal{M}}(\Omega, Z).$$

In fact, (13) is true with $\zeta_M \det(\Omega)^{\frac{1}{2}}$ replaced by $\det \mathcal{M}^{\frac{g}{2}} \cdot \det(\frac{\Omega}{i})^{\frac{1}{2}}$, where the branch of the square root is used which has positive value when Ω is pure imaginary. If f is a smooth function on $\mathbb{R}^{(h,g)}$, going to zero fast enough at ∞, then $\hat{f}$ its Fourier transform is

$$\hat{f}(\zeta) = \int_{\mathbb{R}^{(h,g)}} f(X)\exp\{2\pi i\sigma(\mathcal{M}\zeta{}^tX))\}dX_{11}\cdots dX_{hg}.$$

By the Poission summation formula([I],p.44)

$$\sum_{N\in\mathbb{Z}^{(h,g)}} f(N) = \sum_{N\in\mathbb{Z}^{(h,g)}} \hat{f}(N)$$

we apply this with $f(X) = \exp\{\pi i\sigma(\mathcal{M}(X\Omega{}^tX + 2X{}^tZ))\}$. Then

$$\sum_{N\in\mathbb{Z}^{(h,g)}} f(N) = \vartheta^{\mathcal{M}}(\Omega, Z).$$

To calculate $\hat{f}$, we need the following integral :

Lemma 3.4. *For all* $\Omega \in \mathcal{H}_g,\ Z \in \mathbb{C}^{(h,g)}$,

$$\int_{\mathbb{R}^{(h,g)}} \exp\{\pi i\sigma(\mathcal{M}(X\Omega{}^tX + 2X{}^tZ))\}dX_{11}\cdots dX_{hg}$$
$$= \det \mathcal{M}^{-\frac{g}{2}} \cdot \det(\frac{\Omega}{i})^{-\frac{1}{2}} \exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}.$$

Proof.

$$\int_{\mathbb{R}^{(h,g)}} \exp\{\pi i\sigma(\mathcal{M}(X\Omega{}^tX + 2X{}^tZ))\}dX_{11}\cdots dX_{hg}$$
$$= \exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\cdot$$
$$\int_{\mathbb{R}^{(h,g)}} \exp\{\pi i\sigma(\mathcal{M}(X + Z\Omega^{-1})\Omega{}^t(X + Z\Omega^{-1}))\}dX_{11}\cdots dX_{hg}.$$

As both sides of the equality to be proved are holomorphic in Ω and Z, it suffices to prove they are equal when Ω and Z are pure imaginary matrices. Therefore we may assume

$$\begin{aligned}\Omega =& iA^tA, \quad \text{A is real positive definite symmetric}\\ Z =& iY, \qquad Y \in \mathbb{R}^{(h,g)}.\end{aligned}$$

Then the integral becomes :

$$\exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\cdot\int_{\mathbb{R}^{(h,g)}}\exp\{-\pi\sigma(\mathcal{M}(X + Y(A^tA)^{-1}A^tA^t(X+Y(A^tA)^{-1}))\}dX_{11}\cdots dX_{hg}.$$

Replacing X by $X+Y(A^tA)^{-1}$, this becomes

$$\exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\int_{\mathbb{R}^{(h,g)}}\exp\{-\pi\sigma(\mathcal{M}XA^tA^tX)\}dX_{11}\cdots dX_{hg}.$$

Substituting $W=\sqrt{\mathcal{M}}XA$, the above becomes

$$\begin{aligned}&\exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\\ &\int_{\mathbb{R}^{(h,g)}}\exp\{-\pi\sigma(W^tW)\}(\det\sqrt{\mathcal{M}})^{-g}(\det A)^{-1}dW_{11}\cdots dW_{hg}\\ =&\exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)[\det(A^tA)]^{-\frac{1}{2}}(\det\mathcal{M})^{-\frac{g}{2}}\Pi_{l=1}^{h}\Pi_{j=1}^{g}\\ &\qquad\int_{-\infty}^{\infty}\exp(-\pi W_{lj})dW_{lj}\\ =&\exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\det(\frac{\Omega}{i})^{-\frac{1}{2}}(\det\mathcal{M})^{-\frac{g}{2}}.\end{aligned}$$

We may now calculate $\hat{f}$:

$$\begin{aligned}\hat{f}(\zeta)=&\int_{\mathbb{R}^{(h,g)}}\exp\{\pi i\sigma(\mathcal{M}(X\Omega^tX+2X^tZ))\}\\ &\qquad\exp\{2\pi i\sigma(\mathcal{M}\zeta^tX)\}dX_{11}\cdots dX_{hg}\\ =&\int_{\mathbb{R}^{(h,g)}}\exp\{\pi i\sigma(\mathcal{M}((X\Omega^tX)+2X^t(Z+\zeta)))\}dX_{11}\cdots dX_{hg}\\ =&\det(\mathcal{M})^{-\frac{g}{2}}\det(\frac{\Omega}{i})^{-\frac{1}{2}}\exp\{-\pi i\sigma(\mathcal{M}(Z+\zeta)\Omega^{-1}\,{}^t(Z+\zeta))\}.\end{aligned}$$

Therefore

$$\sum_{N\in Z^{(h,g)}} \hat{f}(N)$$

$$= (\det \mathcal{M})^{-\frac{g}{2}} \det(\frac{\Omega}{i})^{-\frac{1}{2}}$$

$$\sum_{N\in Z^{(h,g)}} \exp\{-\pi i\sigma(\mathcal{M}(Z+N)\Omega^{-1}\,{}^t(Z+N))\}$$

$$= (\det \mathcal{M})^{-\frac{g}{2}} \det(\frac{\Omega}{i})^{-\frac{1}{2}} \exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}$$

$$\sum_{N\in Z^{(h,g)}} \exp\{\pi i\sigma(\mathcal{M}(-N)(-\Omega^{-1})(-N)+2\mathcal{M}(-N)^{-1}(Z\Omega^{-1}))\}$$

$$= (\det \mathcal{M})^{-\frac{g}{2}} \det(\frac{\Omega}{i})^{-\frac{1}{2}} \exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\vartheta^{\mathcal{M}}(-\Omega^{-1}, Z\Omega^{-1}).$$

Thus

$$\vartheta^{\mathcal{M}}(\Omega, Z) = \sum_{N\in Z^{(h,g)}} f(N) = \sum_{N\in Z^{(h,g)}} \hat{f}(N)$$

$$=(\det \mathcal{M})^{-\frac{g}{2}} \det(\frac{\Omega}{i})^{-\frac{1}{2}} \exp\{-\pi i\sigma(\mathcal{M}Z\Omega^{-1}\,{}^tZ)\}\cdot$$

$$\vartheta^{\mathcal{M}}(-\Omega^{-1}, Z\Omega^{-1}).$$

This is the formula (13). This completes the proof of the functional equation (6).

Lemma 3.5. *If $M \in \Gamma_g^{(4)}$, i.e., $M \equiv E_{2g} \mod 4$, then $\zeta_M = \pm 1$ in the formula (6).*

Proof. In fact, $\Gamma_g^{(4)}$ is contained in the group generated by matrices

$$\begin{pmatrix} E & 2B \\ 0 & E \end{pmatrix}, \quad \begin{pmatrix} E & 0 \\ 2C & E \end{pmatrix}, \quad \text{where } B, C \text{ are symmetric matrices.[M]}$$

Case I. $M = \begin{pmatrix} E & 2B \\ 0 & E \end{pmatrix}$, B is symmetric matrix.

The formula (6) reduces to :

$$\vartheta^{\mathcal{M}}(\Omega + 2B, Z) = \zeta_M \vartheta^{\mathcal{M}}(\Omega, Z).$$

Here we may take $\zeta_M = 1$, because in fact,

$$\begin{aligned}
&\vartheta^{\mathcal{M}}(\Omega + 2B, Z) \\
&= \sum_{N \in \mathbb{Z}^{(h,g)}} \exp\{\pi i \sigma(\mathcal{M}N(\Omega + 2B)^t N + 2\mathcal{M}Z^t N)\} \\
&= \sum_{N} \exp\{2\pi i \sigma(\mathcal{M}NB^t N)\} \exp\{\pi i \sigma(\mathcal{M}(N\Omega^t N + 2Z^t N))\} \\
&= \vartheta^{\mathcal{M}}(\Omega, Z).
\end{aligned}$$

Case II. $M = \begin{pmatrix} E & 0 \\ 2C & E \end{pmatrix}$, C is symmetric matrix.
Take $C = -B$. Then,

$$\begin{pmatrix} E & 0 \\ -2B & E \end{pmatrix} = \begin{pmatrix} 0 & -E \\ E & 0^{-1} \end{pmatrix} \begin{pmatrix} E & 2B \\ 0 & E \end{pmatrix} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

So the 8th root of unity ζ_M involved in the formula (6) for $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ cancels out, and we have $\zeta_M = \pm 1$ in the equation for $\begin{pmatrix} E & 0 \\ -2B & E \end{pmatrix}$ (we cannot say $\zeta_M = 1$ unless the appropriate branch of $\det(C\Omega + D)^{\frac{1}{2}}$ is chosen).

Remark 3.6. *$\vartheta^{\mathcal{M}}_{0,0}(\Omega, 0)^2$ is a modular form of weight h and level $\Gamma_g^{(4)}$.*

Proof. In formula (6) we take $Z = 0$. Then

$$\vartheta^{\mathcal{M}}(M < \Omega >, 0)^2 = \zeta_M^2 \det(C\Omega + D)\vartheta^{\mathcal{M}}(\Omega, 0)^2.$$

By lemma 3.5, if $M \in \Gamma_g^{(4)}$, then $\zeta_M = \pm 1$. This completes the proof.

Theorem 3.7. *Let n be even integer. Then for all $X_1, X_2, Y_1, Y_2 \in \frac{1}{n}\mathbb{Z}^{(h,g)}$,*

$$\vartheta^{\mathcal{M}} \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} (\Omega, 0) \cdot \vartheta^{\mathcal{M}} \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix} (\Omega, 0)$$

is a modular form of weight 1 and level $\Gamma_g^{(n^2,2n^2)}$.

Proof. By lemma 3.2,

$$\vartheta^{\mathcal{M}}\begin{bmatrix} X_1\,{}^tD - Y_1\,{}^tC \\ -X_1\,{}^tB + Y_1\,{}^tA\end{bmatrix}(M<\Omega>,0)\cdot\vartheta^{\mathcal{M}}\begin{bmatrix} X_2\,{}^tD - Y_2\,{}^tC \\ -X_2\,{}^tB + Y_2\,{}^tA\end{bmatrix}$$
$$(M<\Omega>,0)$$
$$=\zeta_M^2\det(C\Omega+D)\exp\{-\pi i\sigma(\mathcal{M}(X_1\,{}^tBD{}^tX_1 - 2X_1\,{}^tBC{}^tY_1$$
$$+Y_1\,{}^tAC{}^tY_1))\}\cdot\exp\{-\pi i\sigma(\mathcal{M}(X_2\,{}^tBD{}^tX_2 - 2X_2\,{}^tBC{}^tY_2$$
$$+Y_2\,{}^tAC{}^tY_2))\}\cdot\vartheta^{\mathcal{M}}\begin{bmatrix} X_1 \\ Y_1\end{bmatrix}(\Omega,0)\cdot\vartheta^{\mathcal{M}}\begin{bmatrix} X_2 \\ Y_2\end{bmatrix}(\Omega,0).$$

Since n is even, $\Gamma_g^{(n^2,2n^2)}\subset\Gamma_g^{(4)}$. Thus $\zeta_M^2=1$. If $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in\Gamma_g^{(n^2,2n^2)}$, then $A\equiv E_g \mod n^2$, $D\equiv E_g \mod n^2$, $B\equiv 0 \mod n^2$, and $C\equiv 0 \mod n^2$, and $2n^2$ divides the diagonals of B and C, and ${}^tAC$, ${}^tBD$ are symmetric, thus

$$\sigma(X_1\,{}^tBD{}^tX_1),\sigma(Y_1\,{}^tAC{}^tY_1),\sigma(X_2\,{}^tBD{}^tX_2),\ \text{and}\ \sigma(Y_2\,{}^tAC{}^tY_2)$$

are in $2\mathbb{Z}$.

Hence the exponential factors are equal to 1.

On the other hand, $M\in\Gamma_g^{(n^2,2n^2)}$ lead us that

$$\vartheta^{\mathcal{M}}\begin{bmatrix} X_1^tD - Y_1^tC \\ -X_1^tB + Y_1^tA\end{bmatrix}(M<\Omega>,0)=\vartheta^{\mathcal{M}}\begin{bmatrix} X_1 \\ Y_1\end{bmatrix}(M<\Omega>,0),$$

and

$$\vartheta^{\mathcal{M}}\begin{bmatrix} X_2^tD - Y_2^tC \\ -X_2^tB + Y_2^tA\end{bmatrix}(M<\Omega>,0)=\vartheta^{\mathcal{M}}\begin{bmatrix} X_2 \\ Y_2\end{bmatrix}(M<\Omega>,0),$$

by the definition of theta function. This completes the proof.

4. Mixed theta series

Definition 4.1. A function $\Phi:\mathbb{C}^{(h,g)}\times\mathbb{C}^{(h,g)}\longrightarrow\mathbb{C}$ is called an ***auxiliary theta series of level*** $\mathcal{M}$ with respect to $\Omega\in\mathcal{H}_g$ if it satisfies the following conditions:

(i) $\Phi(U,Z)$ is a polynomial in Z whose coefficients are entire functions,

(ii) $\Phi(U+\lambda, Z+\lambda\Omega+\mu) = e^{-\pi i(\mathcal{M}(\lambda\Omega^t\lambda+2\lambda^t Z))}\Phi(U,Z)$ for all $\lambda, \mu \in \mathbb{Z}^{(h,g)}$.

The space $\Theta_\Omega^{(\mathcal{M})}$ of all auxiliary theta series of level $\mathcal{M}$ with respect to Ω has a basis consisting of the following functions:

$$\vartheta_J^{(\mathcal{M})}\begin{bmatrix} S \\ 0 \end{bmatrix}(\Omega|\lambda, \mu+\lambda\Omega):$$
$$= \sum_N \in \mathbb{Z}^{(h,g)}(\lambda+N+S)^J e^{\pi i\sigma(\mathcal{M}((N+A)\Omega^t(N+S)+(\mu+\lambda\Omega)^t(N+S)))}.$$

where S runs over the cosets $\mathcal{M}^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$ and $J \in \mathbb{Z}_{\geq 0}^{(h,g)}$.

Definition 4.2. A real analytic function $\Phi : \mathbb{R}^{(h,g)} \times \mathbb{R}^{(h,g)} \longrightarrow \mathbb{C}$ is called a *mixed theta series of level* $\mathcal{M}$ with respect to $\Omega \in \mathcal{H}$ if Φ satisfies the following conditions:

(i) $\Phi(\lambda, \mu)$ is a polynomial in λ whose coefficients are entire functions in complex variables $Z = \mu + \lambda\Omega \in \mathbb{C}^{(h,g)}$,

(ii) $\Phi(\lambda+\tilde{\lambda}, \mu + \rightarrow \tilde{\mu}) = e^{-\pi i\sigma(\mathcal{M}(\tilde{\lambda}\Omega^t\tilde{\lambda}+2(\mu+\lambda\Omega)^t\tilde{\lambda}))}\Phi(\lambda,\mu)$ for all $\tilde{\lambda}, \tilde{\mu} \in \mathbb{Z}^{(h,g)}$.

If $S \in \mathcal{M}^{-1}\mathbb{Z}^{(h,g)}/\mathbb{Z}^{(h,g)}$ and $J \in \mathbb{Z}_{\geq 0}^{(h,g)}$,

$$\vartheta_J^{(\mathcal{M})}\begin{bmatrix} S \\ 0 \end{bmatrix}(\Omega|\lambda, \mu+\lambda\Omega)$$
$$= \sum_{N\in\mathbb{Z}^{(h,g)}} (\lambda+N+S)^J e^{\pi i\sigma(\mathcal{M}((N+S)\Omega^t(N+S)+2(\mu+\lambda\Omega)^t(N+S)))}$$

is a mixed theta series of level $\mathcal{M}$.

Theorem 4.3. *The mixed theta series* $\vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|\lambda, 0)$ *is a modular form of weight* $\frac{h}{2}$ *and of level* $\mathcal{M}$ *with charater.*

Proof. By the formula (5) in §3 it is trivial.

Corollary 4.4. If $g = h$, the mixed theta series $\vartheta_J^{(\mathcal{M})}\begin{bmatrix} A \\ 0 \end{bmatrix}(\Omega|\lambda, 0)$ is a cusp form if A is non-zero matrix.

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