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AUTOMORPHISMS OF $\mathcal{A}_{2n}^{(S_0)}$

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1. Introduction

The study of reflexive, but not necessarily self-adjoint, algebras of Hilbert space operators has become one of the fastest growing specialties in operator theory. F. Gilfeather and D. Larson discovered the tridiagonal algebras $\mathcal{A}_2, \mathcal{A}_4, \cdots, \mathcal{A}_\infty$[3]. The tridiagonal algebras are the important classes of non-self-adjoint reflexive algebras. Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with an orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Then A is in $\mathcal{A}_{2n}$ if and only if A has the form

$$\begin{pmatrix} * & * & & & & & * \\ & * & & & & & \\ & * & * & * & & & \\ & & & * & & & \\ & & & * & \cdot & & \\ & & & & & \cdot & \\ & & & & & \cdot & * \\ & & & & & & * \end{pmatrix},$$

with respect to the basis $\{e_1, e_2, \cdots, e_{2n}\}$, where all non-starred entries are zero. If we write the given basis in the order $\{e_1, e_3, \cdots, e_{2n-1}, e_2,$

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$e_4, \cdots, e_{2n}\}$, then the above matrix looks like this

$$\begin{pmatrix}
* & & & & & & * & & & & * \\
& * & & & & & * & * & & & \\
& & \cdot & & & & & * & & & \\
& & & \cdot & & & & & \cdot & & \\
& & & & \cdot & & & & & \cdot & \\
& & & & & * & & & & * & * \\
& & & & & & * & & & & \\
& & & & & & & \cdot & & & \\
& & & & & & & & \cdot & & \\
& & & & & & & & & \cdot & \\
& & & & & & & & & & *
\end{pmatrix},$$

where all non-starred entries are zero. The subalgebra of $\mathcal{B}(\mathcal{H})$, the class of all bounded operators acting on $\mathcal{H}$, consisting of these operators was denoted by $\mathcal{A}_{2n}^{(3)}$[6].

Let S_0 be an $n \times n$ matrix with two 1 in each row and each column and 0 elsewhere as entries. Let S be an $n \times n$ matrix. Then $S_0 \preceq S$ means that if the (i, j)-component of S_0 is 0, then the (i, j)-component of S is also 0. Let $\mathcal{A}_{2n}^{(S_0)}$ be the algebra consisting of the operators of the form $\begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$, where D_1 and D_2 are $n \times n$ diagonal matrices and $S_0 \preceq S$. If S_0 is an $n \times n$ matrix whose (i, i)-component is 1 for all $i = 1, 2, \cdots, n$, $(j+1, j)$-component is 1 for all $j = 1, 2, \cdots, n-1$, $(1, n)$-component is 1 and all other components are zero, then $\mathcal{A}_{2n}^{(S_0)} = \mathcal{A}_{2n}^{(3)}$. So the algebra $\mathcal{A}_{2n}^{(3)}$ is the special form of the algebra $\mathcal{A}_{2n}^{(S_0)}$. In this paper we will investigate the necessary and sufficient condition that the automorphisms of $\mathcal{A}_{2n}^{(S_0)}$ are spatially implemented.

First we will introduce the terminologies which are used in this paper. Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. $\mathcal{A}$ is called a self-adjoint algebra provided A^* is in $\mathcal{A}$ for every A in $\mathcal{A}$. Otherwise, $\mathcal{A}$ is called a non-self-adjoint algebra. If $\mathcal{L}$ is a lattice of orthogonal projections acting on $\mathcal{H}$, $Alg\mathcal{L}$ denotes the algebra of all operators acting on $\mathcal{H}$ that leave invariant every orthogonal projections in $\mathcal{L}$. A subspace lattice $\mathcal{L}$ is a strongly closed lattice of orthogonal projections acting on $\mathcal{H}$, containing 0 and I. Dually, if $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$, then $Lat\mathcal{A}$ is the lattice of all orthogonal projections which leave invariant each operator in $\mathcal{A}$. An algebra $\mathcal{A}$ is reflexive if $\mathcal{A} = AlgLat\mathcal{A}$ and a lattice $\mathcal{L}$ is reflexive if $\mathcal{L} = LatAlg\mathcal{L}$. A lattice $\mathcal{L}$ is a commutative subspace lattice, or CSL, if each pair of projections in $\mathcal{L}$ commutes; $Alg\mathcal{L}$ is called a CSL-algebra. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be commutative subspace lattices. By an isomorphism $\phi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ we mean a strictly algebraic isomorphism, that is, a bijective, linear, multiplicative map. An isomorphism $\phi : Alg\mathcal{L}_1 \to Alg\mathcal{L}_2$ is said to be spatially implemented if there is a bounded invetible operator T such that $\varphi(A) = TAT^{-1}$ for all A in $Alg\mathcal{L}_1$. If $x_1, x_2, \cdots, x_n$ are vectors in some Hilbert space, then $[x_1, x_2, \cdots, x_n]$ means the closed subspace generated by the vectors $x_1, x_2, \cdots, x_n$. Let i and j be two nonzero natural numbers. Then E_{ij} is the matrix whose $(i, j)-$component is 1 and all other entries are zero.

2. Automorphisms of $\mathcal{A}_{2n}^{(S_0)}$

Let $\mathcal{H}$ be a $2n$-dimensional complex Hilbert space with a fixed orthonormal basis $\{e_1, e_2, \cdots, e_{2n}\}$. Let E_{ii}, E_{i,i_1} and E_{i,i_2} be in $\mathcal{A}_{2n}^{(S_0)}$ for all $i(1 \le i \le n)$ and $n+1 \le i_1 < i_2 \le 2n$ and let $E_{j^{(1)},j}$, $E_{j^{(2)},j}$ and E_{jj} be in $\mathcal{A}_{2n}^{(S_0)}$ for all $j(n+1 \le j \le 2n)$ and $1 \le j^{(1)} < j^{(2)} \le n$. Let $\mathcal{L}$ be the subspace lattice generated by $\{[e_1], [e_2], \cdots, [e_n], [e_{j^{(1)}}, e_{j^{(2)}}, e_j] : j = n+1, n+2, \cdots, 2n\}$. Then $\mathcal{A}_{2n}^{(S_0)} = Alg\mathcal{L}$ and $\mathcal{A}_{2n}^{(S_0)}$ is reflexive[1]. Before we investigate the general automorphisms $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ we will consider the automorphisms $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ satisfying $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$. Since $E_{ii}E_{i,i_k}E_{i_k,i_k} = E_{i,i_k}$ for all i and $k(1 \le i \le n, 1 \le k \le 2)$, $\rho(E_{i,i_k}) = \rho(E_{ii})\rho(E_{i,i_k})\rho(E_{i_k,i_k}) = E_{ii}\rho(E_{i,i_k})E_{i_k,i_k}$. Hence $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for some nonzero complex number γ_{i,i_k}. From this we have the following theorem.

THEOREM 1. *Let $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism such that $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$. Then there exist $2n$ nonzero complex numbers $\gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ such that $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$.*

Let $\gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ be $2n$ nonzero complex numbers. Define a linear map $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ by $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$ and $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for all i and $k(1 \le i \le n, 1 \le k \le 2)$.

Then clearly ρ is an automorphism. From this we have the following theorem.

THEOREM 2. *If* $\gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ *be* $2n$ *nonzero complex numbers, then there exists an automorphism* $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ *such that* $\rho(E_{pp}) = E_{pp}$ *for all* $p(1 \le p \le 2n)$ *and* $\rho(E_{i,i_k}) = \gamma_{i,i_k} E_{i,i_k}$ *for all* i *and* $k(1 \le i \le n, 1 \le k \le 2)$.

THEOREM 3. *Let* $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ *be an automorphism such that* $\rho(E_{pp}) = E_{pp}$ *for all* $p(1 \le p \le 2n)$ *and let* $\rho(E_{i,i_k}) = \gamma_{i,i_k} E_{i,i_k}, \gamma_{i,i_k} \ne 0$, *for all* $i, k(1 \le i \le n, 1 \le k \le 2)$. *Then* ρ *is spatially implemented by* $T = (t_{uv})$ *if and only if* T *is diagonal and* $\gamma_{i,i_k} = t_{ii} t_{i_k,i_k}^{-1}$ *for all* $i, k(1 \le i \le n, 1 \le k \le 2)$.

Proof. Let $A = (a_{ij})$ be in $\mathcal{A}_{2n}^{(S_0)}$ and $T = \sum_{u=1}^{2n} t_{uu} E_{uu}$. Then $\rho(A)T = TA$. Hence $\rho(A) = TAT^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Conversely, suppose that ρ is spatially implemented by $T = (t_{uv})$. Since $\rho(E_{pp}) = E_{pp}$, $E_{pp}T = TE_{pp}$ for all $p = 1, 2, \cdots, 2n$. Hence $t_{pq} = 0$ for all $p, q(p \ne q)$. Thus T is diagonal. Let $T = \sum_{u=1}^{2n} t_{uu} E_{uu}$ and $\rho(E_{i,i_k}) = \gamma_{i,i_k} E_{i,i_k}$ for all $i, k(1 \le i \le n, 1 \le k \le 2)$. Then

$$\rho(E_{i,i_k})T = (\gamma_{i,i_k} E_{i,i_k})(\sum_{u=1}^{2n} t_{uu} E_{uu}) = \gamma_{i,i_k} t_{i_k,i_k} E_{i,i_k} \text{and}$$

$$TE_{i,i_k} = (\sum_{u=1}^{2n} t_{uu} E_{uu}) E_{i,i_k} = t_{ii} E_{i,i_k}$$

Hence $\gamma_{i,i_k} = t_{ii} t_{i_k,i_k}^{-1}$ for all $i(1 \le i \le n)$ and $k(1 \le k \le 2)$.

THEOREM 4. *Let* $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ *be an automorphism. Then for each* $j(1 \le j \le 2n)$, *either there exist an integer* p *with* $1 \le p \le n$ *and complex numbers* α_{p,p_1} *and* α_{p,p_2} *such that* $\phi(E_{jj}) = E_{pp} + \alpha_{p,p_1} E_{p,p_1} + \alpha_{p,p_2} E_{p,p_2}$ *or there exist an integer* q *with* $n+1 \le q \le 2n$ *and complex numbers* $\beta_{q^{(1)},q}$ *and* $\beta_{q^{(2)},q}$ *such that* $\phi(E_{jj}) = E_{qq} + \beta_{q^{(1)},q} E_{q^{(1)},q} + \beta_{q^{(2)},q} E_{q^{(2)},q}$.

Proof. Let $\phi(E_{jj}) = \begin{pmatrix} D_1 & S \\ 0 & D_2 \end{pmatrix}$ be in $\mathcal{A}_{2n}^{(S_0)}$. Since $\phi(E_{jj})^2 = \phi(E_{jj})$, we have $D_1^2 = D_1, D_2^2 = D_2$ and $D_1 S + SD_2 = S$. Hence each diagonal element of $\phi(E_{jj})$ is 1 or 0. Since all diagonal elements of $\phi(E_{jj})$ is 0 implies $\phi(E_{jj})^2 = 0$, the (p,p)-component of $\phi(E_{jj})$ is 1 for some $p(1 \le p \le 2n)$. If the (r,r)-component of $\phi(E_{jj})$ is 1 for some r such that $r \neq p$ and $1 \le r \le 2n$, then the (p,p)-component and the (r,r)-component of $\phi(E_{jj})$ are 1. So there exists k with $1 \le k \le 2n$ and $j \neq k$ such that one of the (p,p)-component or the (r,r)-component of $\phi(E_{kk})$ is 1, and so $0 = \phi(E_{jj}E_{kk}) = \phi(E_{jj})\phi(E_{kk}) \neq 0$ which is a contradiction. Hence the (p,p)-component of $\phi(E_{jj})$ is 1 for one and only one $p(1 \le p \le 2n)$. If the (p,p)-component of $\phi(E_{jj})$ is 1 for some p with $1 \le p \le n$, then $\phi(E_{jj}) = E_{pp} + \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$. So $\phi(E_{jj}) = \phi(E_{jj})^2 = E_{pp} + \alpha_{p,p_1}E_{p,p_1} + \alpha_{p,p_2}E_{p,p_2}$ for some complex numbers α_{p,p_1} and α_{p,p_2}. If the (q,q)-component of $\phi(E_{jj})$ is 1 for some q with $n+1 \le q \le 2n$, then $\phi(E_{jj}) = E_{qq} + \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$. So $\phi(E_{jj}) = \phi(E_{jj})^2 = E_{qq} + \beta_{q^{(1)},q}E_{q^{(1)},q} + \beta_{q^{(2)},q}E_{q^{(2)},q}$ for some complex numbers $\beta_{q^{(1)},q}$ and $\beta_{q^{(2)},q}$.

THEOREM 5. *Let $\phi \cdot \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism*
(1) If $1 \le i \le n$ and the (p,p)-component of $\phi(E_{ii})$ is 1, then $1 \le p \le n$
(2) If $n+1 \le j \le 2n$ and the (q,q)-component of $\phi(E_{jj})$ is 1, then $n+1 \le q \le 2n$.

Proof. (1) Suppose that $n+1 \le p \le 2n$. Then $\phi(E_{ii}) = E_{pp} + \alpha_{p^{(1)},p}E_{p^{(1)},p} + \alpha_{p^{(2)},p}E_{p^{(2)},p}$. Let $\phi(E_{i,i_1}) = (\gamma_{uv})$ be in $\mathcal{A}_{2n}^{(S_0)}$. Then

$$\begin{aligned}\phi(E_{i,i_1}) &= \phi(E_{ii})\phi(E_{i,i_1})\phi(E_{i_1,i_1}) \\ &= (E_{pp} + \alpha_{p^{(1)},p}E_{p^{(1)},p} + \alpha_{p^{(2)},p}E_{p^{(2)},p})\phi(E_{i,i_1})\phi(E_{i_1,i_1}) \\ &= (\gamma_{pp}E_{pp} + \alpha_{p^{(1)},p}\gamma_{pp}E_{p^{(1)},p} + \alpha_{p^{(2)},p}\gamma_{pp}E_{p^{(2)},p})\phi(E_{i_1,i_1}).\end{aligned}$$

Since the (p,p)-component of $\phi(E_{i_1,i_1})$ is 0, $\phi(E_{i,i_1}) = 0$. It is a contradiction. Hence $1 \le p \le n$.
(2) By similar proof of (1), (2) holds.

THEOREM 6. *Let $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism and let $1 \le i \le n$. If*

$$\phi(E_{ii}) = E_{pp} + \alpha_{p,p_1}E_{p,p_1} + \alpha_{p,p_2}E_{p,p_2} \text{ and}$$
$$\phi(E_{i_k,i_k}) = E_{qq} + \beta_{q^{(1)},q}E_{q^{(1)},q} + \beta_{q^{(2)},q}E_{q^{(2)},q} \text{ for } k = 1 \text{ or } 2,$$

then there exists a nonzero complex number γ_{pq} such that $\phi(E_{i,i_k}) = \gamma_{pq}E_{pq}$, and $\beta_{pq} = -\alpha_{pq}$.

Proof. Since $1 \le i \le n$, we have $n+1 \le i_k \le 2n$. Hence $1 \le p \le n$ and $n+1 \le q \le 2n$. Let $\phi(E_{i,i_k}) = \sum_{p=1}^{2n} \gamma_{pp}E_{pp} + \sum_{i=1}^{n} \gamma_{i,i_1}E_{i,i_1} + \sum_{i=1}^{n} \gamma_{i,i_2}E_{i,i_2}$. Then

$$\begin{aligned}\phi(E_{i,i_k}) &= \phi(E_{ii})\phi(E_{i,i_k})\phi(E_{i_k,i_k})\\ &= (E_{pp} + \alpha_{p,p_1}E_{p,p_1} + \alpha_{p,p_2}E_{p,p_2})\phi(E_{i,i_k})\phi(E_{i_k,i_k})\\ &= (\gamma_{pp}E_{pp} + \lambda_1 E_{p,p_1} + \lambda_2 E_{p,p_2})(E_{qq} + \beta_{q^{(1)},q}E_{q^{(1)},q} + \beta_{q^{(2)},q}E_{q^{(2)},q})\end{aligned}$$

where $\lambda_1 = \gamma_{p,p_1} + \alpha_{p,p_1}\gamma_{p_1,p_1}, \lambda_2 = \gamma_{p,p_2} + \alpha_{p,p_2}\gamma_{p_2,p_2}$. So every component of $\phi(E_{i,i_k})$ is 0 except the (p,q)-component. Hence $\gamma_{pp} = 0$ for all $p(1 \le p \le 2n)$. Since $\phi(E_{i,i_k}) \ne 0$, we have $\phi(E_{i,i_k}) = \gamma_{pq}E_{pq}$. Since $\gamma_{pq} \ne 0$, either $p_1 = q$ or $p_2 = q$ and either $q^{(1)} = p$ or $q^{(2)} = p$. Let $A = E_{ii} + E_{i,i_k} + E_{i_k,i_k}$. Then $A^2 = E_{ii} + 2E_{i,i_k} + E_{i_k,i_k}$. Since $\phi(A^2) = \phi(A)^2$, the (p,q)-components of $\phi(A)^2$ and $\phi(A^2)$ are equal. So $\alpha_{pq} + \beta_{pq} + 2\gamma_{pq} = 2(\alpha_{pq} + \beta_{pq} + \gamma_{pq})$. Hence $\alpha_{pq} = -\beta_{pq}$.

From Theorem 6, we have the following theorem.

THEOREM 7. *Let $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism. If E_{pq} is in $\mathcal{A}_{2n}^{(S_0)}$ with $p \ne q$, then there exist i and $i_k(1 \le i \le n, 1 \le k \le 2)$ such that $\phi(E_{ii}) = E_{pp} + \alpha_{pq}E_{pq} + \alpha_{pq'}E_{pq'}$ and $\phi(E_{i_k,i_k}) = E_{qq} + \beta_{pq}E_{pq} + \beta_{p'q}E_{p'q}$ for some complex numbers $\alpha_{pq}, \alpha_{p,q'}, \beta_{pq}$ and $\beta_{p'q}$ and there exists a nonzero complex number γ_{pq} such that $\phi(E_{i,i_k}) = \gamma_{pq}E_{pq}$. Moreover $\alpha_{pq} = -\beta_{pq}$.*

From Theorem 7, we have the following theorem.

THEOREM 8. *Let $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism such that the (p,p)-component of $\varphi(E_{pp})$ is 1 for all $p(1 \le p \le 2n)$. Then*
(1) for eich $i(1 \le i \le n)$, $\varphi(E_{ii}) = E_{ii} + \alpha_{i,i_1}E_{i,i_1} + \alpha_{i,i_2}E_{i,i_2}$ for some complex numbers α_{i,i_1} and α_{i,i_2}.
(2) for each $j(n+1 \le j \le 2n)$, $\varphi(E_{jj}) = E_{jj} - \alpha_{j^{(1)},j}E_{j^{(1)},j} - \alpha_{j^{(2)},j}E_{j^{(2)},j}$ for some complex numbers $\alpha_{j^{(1)},j}$ and $\alpha_{j^{(2)},j}$.
(3) for each $E_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$, $\varphi(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for some nonzero complex number γ_{i,i_k}.

THEOREM 9. *Let $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism such that the (p,p)-component of $\varphi(E_{pp})$ is 1 for all $p(1 \le p \le 2n)$. Then there exists an operator T in $\mathcal{A}_{2n}^{(S_0)}$ such that $\varphi(A) = T\rho(A)T^{-1}$ for* *all A in $\mathcal{A}_{2n}^{(S_0)}$, where $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ is an automorphism such that $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$.*

Proof. Let $\varphi(E_{ii}) = E_{ii} + \alpha_{i,i_1}E_{i,i_1} + \alpha_{i,i_2}E_{i,i_2}$ for all $i(1 \le i \le n)$ and $\varphi(E_{jj}) = E_{jj} - \alpha_{j^{(1)},j}E_{j^{(1)},j} - \alpha_{j^{(2)},j}E_{j^{(2)},j}$ for all $j(n+1 \le j \le 2n)$. Then there exist $2n$ nonzero complex numbers $\gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ such that $\varphi(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$. Define an isomorphism $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ by $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$ and $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for all $i(1 \le i \le n)$ and $k(1 \le k \le 2)$. Let $T = \sum_{p=1}^{2n} E_{pp} - \sum_{i=1}^{n} \alpha_{i,i_1}E_{i,i_1} - \sum_{i=1}^{n} \alpha_{i,i_2}E_{i,i_2}$. For each $i(1 \le i \le n)$, since $\varphi(E_{ii}) = E_{ii} + \alpha_{i,i_1}E_{i,i_1} + \alpha_{i,i_2}E_{i,i_2}$ for some complex numbers $\alpha_{i,i_1}, \alpha_{i,i_2}$ and $\rho(E_{ii}) = E_{ii}$, we have $\varphi(E_{ii})T = E_{ii} = TE_{ii} = T\rho(E_{ii})$. For each $j(n+1 \le j \le 2n)$, since $\varphi(E_{jj}) = E_{jj} - \alpha_{j^{(1)},j}E_{j^{(1)},j} - \alpha_{j^{(2)},j}E_{j^{(2)},j}$ for some complex numbers $\alpha_{j^{(1)},j}, \alpha_{j^{(2)},j}$ and $\rho(E_{jj}) = E_{jj}$, we have $\varphi(E_{jj})T = E_{jj} - \alpha_{j^{(1)},j}E_{j^{(1)},j} - \alpha_{j^{(2)},j}E_{j^{(2)},j} = TE_{jj} = T\rho(E_{jj})$. For each $i, k(1 \le i \le n, 1 \le k \le 2)$, since $\varphi(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k} = \rho(E_{i,i_k})$, we have $\varphi(E_{i,i_k})T = (\gamma_{i,i_k}E_{i,i_k})T = \gamma_{i,i_k}E_{i,i_k} = T(\gamma_{i,i_k}E_{i,i_k}) = T\rho(E_{i,i_k})$. Thus $\varphi(A) = T\rho(A)T^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$.

THEOREM 10. *Let $\alpha_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ be $2n$ complex numbers and let $\gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ be $2n$ nonzero complex numbers. Then the linear map $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ defined by*

$$\varphi(E_{ii}) = E_{ii} + \alpha_{i,i_1}E_{i,i_1} + \alpha_{i,i_2}E_{i,i_2} \text{ for all } i(1 \le i \le n),$$
$$\varphi(E_{jj}) = E_{jj} - \alpha_{j^{(1)},j}E_{j^{(1)},j} - \alpha_{j^{(2)},j}E_{j^{(2)},j} \text{ for all } j(n+1 \le j \le 2n),$$
$$\varphi(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k} \text{ for all } i,k(1 \le i \le n, 1 \le k \le 2),$$

is an automorphism.

Proof. Let T be as in Theorem 9. Then $\varphi(E_{pp}) = T\rho(E_{pp})T^{-1}$ for all $p(1 \le p \le 2n)$ and $\varphi(E_{i,i_k}) = T\rho(E_{i,i_k})T^{-1}$ for all $i,k(1 \le i \le n, 1 \le k \le 2)$, where $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ is an automorphism satisfying $\rho(E_{pp}) = E_{pp}$ for all $p(1 \le p \le 2n)$ and $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for all $i(1 \le i \le n)$ and $k(1 \le k \le 2)$. Hence φ is an automorphism

THEOREM 11. *Let $\alpha_{i,i_k}, \gamma_{i,i_k}(1 \le i \le n, 1 \le k \le 2)$ and φ be as in Theorem 10 Let T be as in Theorem 9. Then φ is spatially* *implemented by B if and only if $B = TS$ for some diagonal invertible matrix S satisfying $\gamma_{i,i_k} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$.*

Proof. Suppose that φ is spatially implemented by B. Then $\varphi(B) = BAB^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Let $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism defined by $\rho(E_{jj}) = E_{jj}$ for all $j(1 \leq j \leq 2n)$ and $\rho(E_{i,i_k}) = \gamma_{i,i_k}E_{i,i_k}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$. Since $\varphi(A) = T\rho(A)T^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$, we have $\varphi(A) = T\rho(A)T^{-1} = BAB^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Hence $\rho(A) = T^{-1}BAB^{-1}T = (T^{-1}B)A(T^{-1}B)^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Put $S = T^{-1}B = (s_{uv})$. Then ρ is spatially implemented by S. By Theorem 3, S is diagonal and $\gamma_{i,i_k} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$. Conversely, suppose that $B = TS$ for some diagonal matrix S satisfying $\gamma_{i,i_k} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$. Since S is diagonal and $\gamma_{i,i_k} = s_{ii}s_{i_k,i_k}^{-1}$, ρ is spatially implemented by $S = T^{-1}B$. Hence $\varphi(A) = T\rho(A)T^{-1} = TSAS^{-1}T^{-1} = BAB^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$.

THEOREM 12. *Let $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism. Then there exists a $2n \times 2n$ unitary matrix U and an automorphism $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ with the (p,p)-component of $\varphi(E_{pp})$ is 1 for all $p(1 \leq p \leq 2n)$ such that $\phi(A) = U\varphi(A)U^*$ for all A in $\mathcal{A}_{2n}^{(S_0)}$*

Proof. Let $\sigma = \begin{pmatrix} 1 & 2 & . & . & . & 2n \\ \sigma(1) & \sigma(2) & . & . & . & \sigma(2n) \end{pmatrix}$ be a permutation such that the $(\sigma(i), \sigma(i))$-component of $\phi(E_{ii})$ is 1 for all $i(1 \leq i \leq 2n)$. Let V be $2n \times 2n$ matrix whose $(p, \sigma(p))$−component is 1 for all $p(1 \leq p \leq 2n)$ and all other components are 0. Define $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ by $\varphi(A) = V\phi(A)V^*$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Then by simple calculation, φ is an automorphism and the (p,p)-component of $\varphi(E_{pp})$ is 1 for all $p(1 \leq p \leq 2n)$. Put $U = V^*$. Then $\phi(A) = U\varphi(A)U^*$ for all A in $\mathcal{A}_{2n}^{(S_0)}$.

From Theorems 9 and 12, we have the following theorem.

THEOREM 13. *Let $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism. Then there exist an automorphism $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ satisfying $\rho(E_{pp}) = E_{pp}$ for all $p(1 \leq p \leq 2n)$ and an invertible operator Y such that $\phi(A) = Y\rho(A)Y^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$.*

THEOREM 14. *Let $\phi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ be an automorphism. Let*

$$\sigma = \begin{pmatrix} 1 & 2 & . & . & . & 2n \\ \sigma(1) & \sigma(2) & . & . & . & \sigma(2n) \end{pmatrix}$$

be a permutation such that the $(\sigma(i), \sigma(i))$-component of $\phi(E_{ii})$ is 1 for all $i(1 \leq i \leq 2n)$ and let $\phi(E_{i,i_k}) = \gamma_{\sigma(i),\sigma(i_k)} E_{\sigma(i),\sigma(i_k)}$ for all E_{i,i_k} in $\mathcal{A}_{2n}^{(S_0)}$. Then ϕ is spatially implemented by R if and only if $R = UTS$ for some diagonal invertible matrix $S = (s_{uv})$ satisfying $\gamma_{\sigma(i),\sigma(i_k)} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$, where $U^* = \sum_{p=1}^{2n} E_{p,\sigma(p)}$ and $T = \sum_{p=1}^{2n} E_{pp} - \sum_{i=1}^{n} \alpha_{i,i_1} E_{i,i_1} - \sum_{i=1}^{n} \alpha_{i,i_2} E_{i,i_2}$.

Proof. Note that the $(\sigma(p), \sigma(p))$-component of $\phi(E_{pp})$ is 1 for all $p(1 \leq p \leq 2n)$. By Theorem 6, $\phi(E_{i,i_k}) = \gamma_{\sigma(i),\sigma(i_k)} E_{\sigma(i),\sigma(i_k)}$ for some nonzero complex number $\gamma_{\sigma(i),\sigma(i_k)}$. From Theorem 12, there exists an automorphism $\varphi : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ satisfying the (p,p)-component of $\varphi(E_{pp})$ is 1 for all $p(1 \leq p \leq 2n)$ such that $\phi(A) = U\varphi(A)U^*$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Hence

$$\begin{aligned} \varphi(E_{i,i_k}) &= U^*\phi(E_{i,i_k})U \\ &= \Big(\sum_{p=1}^{2n} E_{p,\sigma(p)}\Big)\big(\gamma_{\sigma(i),\sigma(i_k)} E_{\sigma(i),\sigma(i_k)}\big)\Big(\sum_{p=1}^{2n} E_{\sigma(p),p}\Big). \\ &= \gamma_{\sigma(i),\sigma(i_k)} E_{i,i_k} \end{aligned}$$

Define an automorphism $\rho : \mathcal{A}_{2n}^{(S_0)} \to \mathcal{A}_{2n}^{(S_0)}$ by $\rho(E_{pp}) = E_{pp}$ for all $p(1 \leq p \leq 2n)$ and $\rho(E_{i,i_k}) = \gamma_{\sigma(i),\sigma(i_k)} E_{i,i_k}$ for all $i(1 \leq i \leq n)$ and $k(1 \leq k \leq 2)$. Then from Theorem 9, $\varphi(A) = T\rho(A)T^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. Suppose that ϕ is spatially implemented by R. Then $\phi(A) = RAR^{-1}$ for all A in $\mathcal{A}_{2n}^{(S_0)}$. So $\rho(A) = T^{-1}\varphi(A)T = T^{-1}U^*\phi(A)UT = (UT)^{-1}RAR^{-1}(UT)$. Put $S = (UT)^{-1}R = (s_{uv})$. Then ρ is spatially implemented by S. By Theorem 3, S is diagonal and $\gamma_{\sigma(i),\sigma(i_k)} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq k \leq 2)$. Conversely, suppose that $R = UTS$ for some invertible diagonal matrix $S = (s_{uv})$ satisfying $\gamma_{\sigma(i),\sigma(i_k)} = s_{ii}s_{i_k,i_k}^{-1}$ for all i and $k(1 \leq i \leq n, 1 \leq k \leq 2)$. Since S is diagonal and $\gamma_{\sigma(i),\sigma(i_k)} = s_{ii}s_{i_k,i_k}^{-1}$ for all $i, k(1 \leq i \leq n, 1 \leq$

$k \leq 2$), ρ is spatially implemented by S. Hence $\phi(A) = U\varphi(A)U^* = UT\rho(A)T^{-1}U^* = UTSAS^{-1}T^{-1}U^*$. Hence ϕ is spatially implemented by $R = UTS$.

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