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THE CONLEY INDEX ON THE MORSE THEORY

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1. Introduction

The idea of the Conley index goes back to Morse and Smale. Morse showed that the topology of the sublevel sets of a smooth function changed when the level moved through a critical level and therby proved his inequalities relating the Betti numbers of the manifold and the indices of the critical point. Smale[15] defined certain dynamical systems, showed they omitted 'filtrations 'analogous to the filtration of a manifold by the sublevel sets of a Morse function and therby extended Morse theory from gradient dynamical systems to a vastly more general class.

The homology index developed over the years by C.C. Conley and his students has been justly termed the Conley index.

The basic theory of the Conley index may be summerized as follows:

(1) The Conley index of an isolated invariant set of a flow is independent of the index pair used to define it.

(2) The Conley index is invariant under continuation.

In this paper we shall give an exposition of Conley index including the necessary background on the Morse index

In a prelimanary section we shall briefly describe the classical Morse inequalities and give a proof which is based on the Conley index.

The main results of this paper are as follows: If RP^n is the real projective n-space and $f\colon RP^n \to R$ is a differentiable function on RP^n with nondegenerate critical points, then the Morse function on RP^n have $n+1$ critical points. And the non-orientable case RP^2 having 3 critical points and the orientable case RP^3 having 4 critical points determine a chain complex which represents a special case of Conley's connection matrix.

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2. Morse inequalities and the Conley index

On n-dimensional Riemannian manifold M we consider the gradient flow

$$(2.1) \qquad \dot{x} = -\nabla f(x)$$

of a smooth function $f\colon RP^n \to R$. The flow of (2.1) will be denoted by

$$\frac{d}{dt}\phi^t = -\nabla f \circ \phi^t, \qquad \phi^0 = 1$$

PROPERTIES. *(1) f decrıse along the orbits of ϕ^t.*

(2) The rest points of ϕ^t are the critical points of f.

(3) The rest points of ϕ^t are hyperbolic ıf f is a Morse function meaning that the Hessian of f is nonsıngular at every point.

DEFINITION. *The unstable set*

$$W^u(x) \equiv \{p \in M | \lim_{t \to -\infty} \phi^t(p) = x\}$$

The stable set

$$W^s(x) \equiv \{p \in M | \lim_{t \to \infty} \phi^t(x) = x\}$$

for each critical point x.

In this case the unstable set and the stable set are submanffold of M. In fact, $W^u(x)$ is a C^∞ manifold diffeomorphic to R^k, where $ind(x) = k$, hence $ind(x) = dimW^u(x)$

Simiarly, $dimW^s(x) = n - k$, where $ind(x) = k, dimM = n$.

Throughout the paper H_* will denote singular homology.

THEOREM 2.2. (M. MORSE). *Let*

$$c_k - c_{k-1} + \cdots \pm c_0 \geq \beta_k - \beta_{k-1} + \cdots \pm \beta_0$$

for $k = 0, 1, \ldots, n$ and equality holds $k = n$,where c_k: the number of critical points of index k. β_k: the Betti numbers, $\beta_k = rankH_k(M; R)$ of the manifold M for any principal ideal domain R.

DEFINITION. *(1) A set $S \subset M$ is called invariant if $\phi^t(S) = S$ for every $t \in R$.*

(2) S is called isolated if there exists a neighborhood N of S such that

$$S = I(N) = \bigcap_{t \in R} \phi^t(N)$$

(3) An index pair for an isolated invariant set $S \subset M$ is a pair of compact $L \subset N$ such that $S = I(cl(N \setminus L)) \subset int(N \setminus L)$ and

(i) $x \in L, \phi^{[0,t]}(x) \subset N \Rightarrow \phi^t(x) \in L$

(ii) $x \in N \setminus L \Rightarrow \exists t > 0$ with $\phi^{[0,t]} \subset N$.

Condation (i) says that L is positively invariant in N and (ii) means that every orbit which leaves N goes through L first.

LEMMA2.2. *(1) Every isolated invariant set S admits an index pair such that the topological quotient N/L has the homotopy type of a ~~finite polyhedron~~.*

(2) The homotopy type of N/L is independent of the choice of the index pair.

Proof. See [2] and [11].

LEMMA2.3. (C.C.CONLEY). *If (N_α, L_α) and (N_β, L_β) are two index pair for S then the index spaces N_α/L_α and N_β/L_β are homotopy equivalent.*

The Conley index of S is the homotopy type of the pointed space N/L. If L is a neighborhood deformation retract in N then the homology of the index spaces N/L agrees with the homology of the pair N, L and is characterized by the index polynomial

$$P_S(t) = \sum_k rank H_k(M^b, M^a; R)t^k.$$

The Conley index is additive in the sense that $P_S(t) = P_{s_1}(t) + P_{S_2}(t)$ whenever S is the disjoint union of the isolated invariant sets S_1 and S_2. It follows from Lemma 2.3 and the additivity of the Conley index that

$$\sum_k rank H_k(M^b, M^a; R)t^k = \sum_{x \in S} t^{ind(x)} \tag{2.2}$$

where $M^a = \{x \in M | f(x) \leq a\}$ and $a < b$.

Proof of Theorem 2.1. Define $\beta_k^a = rank H_k(M^a; \Re)$ and let c_k^a be the number of the critical points $x \in M$ of f with $ind(x) = k$ and $f(x) \leq a$. If $a < b$ are regular values of f such that c is the only critical value of f in the interval (a , b) then it follows from (2.2) that $rank H_k(M^b, M^a; \Re) = c_k^b - c_k^a$ and hence the homology exact sequence

$$H_{k+1}(M^b, M^a; \Re) \xrightarrow{\partial_k} H_k(M^a; \Re) \rightarrow H_k(M^b; \Re) \rightarrow H_k(M^b, M^a; \Re)$$

shows that $rank\partial_{k-1} + rank\partial_k = c_k^b - c_k^a - \beta_k^b + \beta_k^a$. Equivalently $P_f^b(t) - P_M^b(t) = P_f^a(t) - P_M^a(t) + (1+t)Q^{ab}(t)$ where $P_M^a(t) = \sum_{k=0}^n \beta_k^a t^k$, $P_f^a(t) = \sum_{k=0}^n c_k^a t^k, Q^{ab}(t) = \sum_{k=0}^n rank\partial_k t^k$.

In Particular,$Q^{ab}(t)$ is a polynomial with nonnegative coefficients. It follows inductively that

$$P_f^a(t) - P_M^a(t) = (1+t)Q^a(t)$$

where $Q^a(t)$ is a polynomial whose nonnegative coefficients are given by $\rho_k^a = \sum_{j=0}^k (c_{k-j}^a - \beta_{k-j}^a) \geq 0$

For $a > supf$ these are the Morse inequalities.

3.Connecting orbits and the Conley's connection matrix

DEFINITION. *The gradient flow ϕ^s of a Morse function $f: M \rightarrow \Re$ is said to be of Morse-Smale type if for any two critical points x and y the stable and unstable manifolds $W^s(x)$ and $W^\mu(y)$ intersect transversally.*

If f is of Morse-Smale type then the connecting orbits determine the following chain complex.

We first choose an orientation of the vector space $E^\mu(x) = T_x W^\mu(x)$ for every critical point of f and denote by $< x >$ the pair consisting of a critical point x and this orientation. For every $k = 0, 1. \ldots, n$ we then denote by C_k the free group

$$C_k = \bigoplus_x Z < x >$$

where x run over all critical points of index k.

In this case one can define an integer n(y,x) by assigning a number +1 or -1 to every connecting orbit and taking the sum. Let $\gamma(s)$ be such a connecting orbit meanning a solution of (2.1) with $\lim_{s\to-\infty}\gamma(s)=y$ and $\lim_{s\to\infty}\gamma(s)$. Then $< y >$ induces an orientation on the orthgonal complement E^u_γ of $v=\lim_{s\to-\infty}|\dot\gamma(s)|^{-1}\dot\gamma(s)$ in $E^u(y)$. In this case $ind(x)=ind(y)-1=k$ the tangent flow induces an isomorphism from $E^u_\gamma(y)$ onto $E^u(x)$ and we define n_γ to +1 or -1 according to whether this map is orientation preserving or reversing. Define $n(y,x)=\sum_\gamma n_\gamma$ where the sum runs over all orbits of (2.1) connecting y to x. Then the boundary operator $\partial^c_k\colon C_{k+1}\to C_k$ of the chain complex is defined by

$$\partial^c < y >=\sum_x n(y,x) < x >$$

where the sum runs over all critical points of index k.

One can extend this chain complex to coefficients in any abelian group G by defining $C_k(G)=G\otimes C_k$ and $\partial^c_k(G)=1_G\otimes\partial^c_k\colon C_{k+1}(G)\to C_k(G)$. The significance of the above construction rests on the following result.

THEOREM 3.1 (R.THOM, S.SMALE, J MILNOR, C.CONLEY).
(i) $\partial^c_{k-1}(G)\circ\partial^c_k(G)=0$
(ii) $H_k(M;G)=\frac{Ker\partial^c_{k-1}(G)}{Im\partial^c_k(G)}$

REMARK. *If $T_xM = E^u(x)\oplus E^s(x)$, the orientation $< x >$ of $E^u(x)=T_xW^u(x)$ induces an orientation of $E^s(x)=T_xW^s(x)$. The nonorientable case can be treated by considering the Z/2-invariant lift of f to the oriented double cover of M*

We shall now descrive Conley's connection matrix for the special case of a Morse-Smale gradient flow. For every critical point *x of f, let N_x, L_x denote the index pair described in section 2 and observe that an orientation of $E^u(x)=T_xW^u(x)$ determines a generator of $H_k(N_x,L_x;Z)\simeq Z$ where $k=ind(x)$. This shows that the group C_k can be identified with*

$$C_k=\oplus_x H_k(N_x,L_x;Z)$$

where the sum runs over all critical points of index k.

Since $H_k(N_x, L_x; Z)$ is a free group it follows from the universal coefficient theorem that the natural homomorphism $G \otimes H_k(N_x, L_x; Z) \to H_k(N_x, L_x; G)$ is an isomorphism and hence

$$G \otimes C_k = \oplus_x H_k(N_x, L_x; G) = C_k(G).$$

REMARK. *(1) If ϕ^t is a Morse-Smale flow then $M(y,x) \equiv W^u(y) \cap W^s(x)$ is a submanifold of M of dimension $ind(y) - ind(x)$.*

(2) If $ind(y) - ind(x) = 1$ then $S(y,x) \equiv M(y,x) \cup \{x, y\}$ is an isolated invariant set.

(3) Let N_2, N_0 be an index pair for S(y,x) and define $N_1 = N_0 \cup (N_2 \cap M^a)$ where $f(x) < a < f(y)$. Then N_2, N_1 is an index pair for y and N_1, N_0 is an index pair for x.

Define the homomorphism

$$\Delta_k(x, y; G) \colon H_{k+1}(N_y, L_y; G) \to H_k(N_x, L_x; G)$$

to be the composition

$$H_{k+1}(N_y, L_y; G) \to H_{k+1}(N_2, L_1; G) \xrightarrow{\partial} H_k(N_1, N_0; G) \to H_k(N_x, L_x; G)$$

where the first and third isomorphism is induced by the flow defined homotopy equivalence of Lemma 2.3. This determines a homomorphism

$$\Delta_k(G) \colon C_{k+1}(G) \to C_k(G)$$

which is a special case of Conley's connection matrix and agrees with the boundary operator of Milnor and Witten.

LEMMA 3.2. $\partial^c(G) = \Delta(G)$

THEOREM 3.3. *If RP^n is the real projective n-space and $f: RP^n \to R$ is a differentiable function on RP^n with nondegenerate critical points, then the Morse function f on $\Re P^n$ have n+1 critical points.*

Proof. We will think of RP^n as equivalence classes of (n+1)-tuples $(x_0, \dots, x_n)$ of real numbers, with $\sum_j^n x_j^2 = 1$. Denote the equivalence class of $(x_0, \dots, x_n)$ by $(x_0 : \cdots : x_n)$. Define a real valued function f on RP^n by the identity $f(x_0 : \cdots : x_n) = \sum_{j=1}^n c_j x_j^2$ where $c_0, \dots, c_n$ are distinct real constants.

In order to determine the critical points of f, consider the following local coordinate system. Let U_0 be the set of $(x_0 : \dots, x_n)$ with $x_0 \neq 0$ i.e. $U_0 = \{(1 : \frac{x_1}{x_0} : \cdots : \frac{x_n}{x_0})\}$. Set $|x_0|\frac{x_j}{x_0} = y_j$. Then $y_1, \dots, y_n : U_0 \to R$ are the required coordinate functions, mapping U_0 diffeomorphically onto the open unit ball in R^n.Clearly $|x_j|^2 = y_j^2$, since $\sum_{j=0}^n x_j^2 = 1, y_0^2 = x_0^2 = 1 - \sum_{j=1}^n x_j^2 = 1 - \sum_{j=1}^n y_j^2$, so that $f = c_0 + \sum_{j=1}^n (c_j - c_0) y_j^2$ throughout the coordinate neighborhood U_0. Thus the only critical point $p_0 = (1 : 0 : \cdots : 0)$ of the cooedinate system, since $f = c_0 + \sum_{j=1}^n (c_j - c_0) y_j^2$,

$$df(x_0 : x_1; \cdots : x_n) = 0 \Leftrightarrow$$
$$(x_0 : x_1 : \cdots : x_n) = (x_0 : 0 : \cdots : 0) = (1 : 0 : \dots ; 0)$$

At this point f *is nondegenerate and has index equal to the number of* j *with* $c_j < c_0$.

Similarly one can consider other coordinate systems centered at the points $p_1 = (0 : 1 : 0 : \cdots : 0), \dots, p_n = (0 : 0 : \cdots : 0 : 1)$. *It follows that* $p_0, \dots, p_n$ *are the only critical points of* f. *The index of* f *at* p_k *is equal to the number of* j *with* $c_j < c_k$. *Thus every possible index between 0 and n occurs exactly once.*

RP^n *has the homotopy type of a CW-complex of the form* $e^0 \cup e^1 \cup \cdots \cup e^n$.

EXAMPLE 1. *On RP^2 (nonorientable case)*

RP^2 *has 3 critical points x, y and z. If* $ind(x) = 2, ind(y) = 1$ *and* $ind(z) = 0$. *In this example the connection matrix is given by*

$$C_2 \xrightarrow{2} C_1 \xrightarrow{0} C_0$$

with $C_k = Z$ *and determines the integral homology* $H_*(RP^2, Z) = (Z, Z/2, 0)$

EXAMPLE 2. *On RP^3 (orientable case)*

RP^3 has 4 critical points x,y,z and w. If $ind(x) = 3, ind(y) = 2, ind(z) = 1$ and $ind(w) = 0$.

$$0 \to C_3 \xrightarrow{0} C_2 \xrightarrow{2} C_1 \xrightarrow{0} C_0 \to 0$$

with $C_k = Z$ and $H_*(RP^3; Z) = (Z, Z/2, 0, Z)$

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