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RIESZ–REPRESENTATION FORMULAR ON EXTENDED HARDY SPACES

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1. Introduction

S.Walters ([15]) examined properties of $H^p(0 < p < 1)$ and showed that H^p is isometric to a closed subspace of L^p. And he conjectured that the dual of H^p is empty except for the zero functional. However, this answer is wrong because $(H^p)^*(0 < p < 1)$ has sufficiently many members to distinguish elements of H^p.

P.L.Duren,W.Romberg and L.Shields([3]) obtained many results on the dual spaces of H^p. In particular, this paper contains the failure of the Hahn-Banach separation theorem as well as an example of a subspace of H^p which has the separation property but does not have the Hahn-Banach extension property. They also construct the extended spaces B^p of which $H^p(0 < p < 1)$ is a dense subset, and investigate the properties of B^p.

In 1982, N.J.Kalton and D.A.Trautman ([7]) gave a number of results on the closed subspace of $H^p(0 < p < 1)$. This result shows that H^p can have no complemented locally convex subspaces; this is the answer to a question of J.H.Shapiro ([10],[11], and [12]). Moreover, they proved that H^p can not have any locally convex subspace with the Hahn-Banach extension property.

In this paper, we find out some properties of H^P and $(H^p)^*$ with $0 < p < 1$, introduce the extended B^p spaces, and investigate the relation between B^p and H^p. Also, we apply the Riesz representation theorem to B^p space.

2. Riesz-representation formular on B^p spaces

In this section, we study the structure and some properties of H^p, for $0 < p < 1$. Also, we extend H^p spaces to larger spaces and apply the properties which are satisfied in H^p to extension of H^p.

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Let D be an open unit disc, T be the unit circle in the complex plane C and $H(D)$ be the set of holomorphic functions in D. For any function f in $H(D)$, we define

$$M_p(f,r) = \{\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\}^{\frac{1}{p}},\ 0 < p < \infty;$$
$$M_\infty(f,r) = sup\{|f(re^{i\theta})| : 0 \le \theta \le 2\pi\};$$
$$||f||_p = \lim_{r\to 1} M_p(f,r),\ 0 < p \le \infty.$$

A function holomorphic f in D is said to be of class H^p $(0 < p \le \infty)$ if $||f||_p$ is finite.

First, we consider the space H^p as a linear space. If each $f \varepsilon H^p$ is identified with its boundary function $\tilde{f}$, H^p can be regarded as a subspace of L^p, $0 < p < \infty$. It is well known that H^p is a Banach space if $1 \le p \le \infty$, but the space H^p is not normable if $0 < p < 1$.

The inequality $(a+b)^p \le a^p + b^p$ for $a \ge 0$, $b \ge 0$ is valid for $0 < p < 1$. This yields the following lemma.

Lemma 2.1. *For any $f, g \varepsilon H^p$, define $d(f,g)$ by*

$$d(f,g) = ||f-g||_p^p$$

then $d(\ ,\ )$ is a metric on H^p.

We recall that for $0 < p < \infty$, H^p is the L^p closure of the set of polynomials in $e^{i\theta}$. So by Lemma 2.1, we obtain the following theorem.

Theorem 2.2. *H^p is a complete metric space.*

We note that $H^p(0 < p < 1)$ is a F-space, in the terminology introduced by Banach. For F-spaces with respect to sequence, see [8], [9], [10], [11], [12], and [13].

For any member f in H^p, we consider the modulus of $f(z)$ $(z \varepsilon D)$ whether bounded or not. Suppose that $f(z)$ is a nonzero function. Then by F.Riesz decomposition theorem, we may write $f(z) = g(z)$ $h(z)$, where $h(z)$ is holomorphic and bounded by unity on D, and $g \varepsilon H^p$ with $||g||_p = ||f||_p$ and $g(z) \neq 0$ on D. Thus it is clear that $[g(z)]^p$

may be defined properly so that it is a member of H^1. By Cauchy's integral formula

$$[g(z)]^p = \frac{1}{2\pi}\int_0^{2\pi} \frac{[g(re^{i\theta})]^p re^{i\theta}}{re^{i\theta}-z} d\theta, \ |z| < r < 1.$$

Thus

$$|g(z)|^p \leq \frac{r}{r-|z|}\left(\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta\right)$$

$$\leq \frac{r}{r-|z|}||g||_p^p = \frac{r}{r-|z|}||f||_p^p.$$

Therefore the following fact holds.

Proposition 2.3. *For each $f \varepsilon H^p$,*

$$|f(z)| \leq \frac{||f||_p}{(1-|z|)^{\frac{1}{p}}}, \ z \,\varepsilon\, D.$$

It is also interesting in the case that the real part of $f(z)$ is positive.

Proposition 2.4. *Every holomorphic function $f(z)$ with positive real part is of class H^p for $0 < p < 1$.*

Proof. Without loss of generality, we can suppose $f(0) = 1$. The range of f is contained in the right half-plane, so f is subordinate to

$$\frac{1+z}{1-z} = p_r(\theta) + iQ_r(\theta)$$

where $p_r(\theta)$ is the Poisson kernel and

$$Q_r(\theta) = \frac{2r\sin\theta}{1-2r\cos\theta+r^2}$$

is the conjugate Poisson kernel. It follows by the Littlewood's subordinate theorem [16] that

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta \leq \int_0^{2\pi} \left|\frac{1+re^{i\theta}}{1-re^{i\theta}}\right|^p d\theta$$

$$\leq \int_0^{2\pi} |Q_1(\theta)|^p d\theta < \infty,$$

for any $p < 1$, since $\frac{(1+z)}{(1-z)}$ is in H^p and $P_1(\theta) = 0$ for $\theta \neq 0$.

For later reference we now list some known results, most of which are due to Hardy and Littlewood.

Theorem 2.5 ([6], p.415). *If $f' \varepsilon H^p$ for some $p \varepsilon (0,1)$, then $f \varepsilon H^q$,* where $q = \frac{p}{(p-1)}$.

Theorem 2.6 ([6], p.412).. *If $f \varepsilon H^p (0 < p < 1)$, then*

$$\int_0^1 (1-r)^{\frac{1}{p}-2} M_1(f,r)dr \leq c||f||_p$$

where c is a constant depending only on p.

Theorem 2.7 ([6], p.408).. *If $f(z) = \sum a_n z^n \varepsilon H^p \, (0 < p \leq 1)$, then*

$$|a_n| \leq c n^{\frac{1}{p}-1} ||f||_p \quad n = 0, 1, 2, \cdots,$$

where c is a constant depending only on p. Furthermore,

$$a_n = o(n^{\frac{1}{p}-1}).$$

As the theory of functional analysis, a linear functional ψ on H^p is said to be bounded (written by $\psi \varepsilon (H^p)^*$) if

$$||\psi|| = \sup_{||f||_p \leq 1} |\psi(f)| < \infty.$$

It follows from the definition that

$$|\psi(f)| \leq ||\psi|| \; ||f||_p$$

for all $f \varepsilon H^p$. It is easily verified that a linear functional on H^p is bounded if and only if it is continuous, and that $(H^p)^*$ is a Banach space. Moreover, the principle of uniform boundedness and the closed graph theorem remain valid for $0 < p < 1$ [1].

If $1 < p < \infty$, it is well known that every bounded linear functional ψ in H^p has a unique representation.

$$\psi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{-i\theta}) d\theta,$$

where $g \varepsilon H^q$, $q = \frac{p}{(p-1)}$. The following may be regarded as an extension of this result to $0 < p < 1$.

Theorem 2.8[2].. *Let $\psi\varepsilon(H^p)^*, 0 < p < 1$. Then there is unique function g such that*

$$\psi(f) = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta})g(e^{-i\theta})d\theta,\ f\varepsilon H^p,$$

where $g(z)$ is holomorphic in D and continuous on $\bar{D}$.

Next we introduce B^p spaces [4] and investigate some properties of this space, finally apply the Theorem 2.8 to this space.

Fix p, $0 < p < 1$. Let B^p denote the space of functions $f(z)$ holomorphic in D for which

$$||f||_{B^p} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f(re^{i\theta})|(1-r)^{\frac{1}{p}-2} dr d\theta < \infty.$$

If we use the quantity $M_p(f,r)$, we can rewrite as following

$$||f||_{B^p} = \int_0^1 (1-r)^{\frac{1}{p}-2} M_1(f,r)dr.$$

It turns out H^p is a subspace of B^p using Theorem 2.7, especially $B^p = H^p$ for $p = \frac{1}{2}$. Thus the space B^p is in some respect "extended" than H^p space. For typographical reasons we shall frequently omit the superscript p in writing norms, $||f||_B$ denote the norm in B^p. The following lemmas are very important to prove one proposition and the last extended theorem.

Lemma 2.9. *For each $f\varepsilon B^p$,*

$$|f(z)| \leq c_p ||f||_B (1-r)^{\frac{-1}{p}},\ z\varepsilon D.$$

Proof.. Let $R < r < 1$, then

$$\begin{aligned} ||f||_B &\geq \int_R^1 (1-r)^{\frac{1}{p}-2} M_1(f,r)dr \\ &\geq M_1(f,R)(\frac{1}{p}-1)^{-1}(1-R)^{\frac{1}{p}-1} \end{aligned}$$

Hence

$$M_1(f,R) \le (\frac{1}{p}-1)||f||_B(1-R)^{1-\frac{1}{p}}.$$

From this, the estimate follows by writing

$$f(z) = \frac{1}{2\pi i}\int_{|\zeta|=R}\frac{f(\zeta)}{\zeta - z}d\zeta,$$

where $R = \frac{1}{2}(1+|z|)$.

Lemma 2.10. *For each $f \varepsilon B^p$, $f_\rho \to f$ in B^p norm as $\rho \to 1$, where $f_\rho(z) = f(\rho z)$.*

Proof. Given $f\varepsilon B^p$ and $\varepsilon > 0$, choose $r < 1$ such that

$$\int_R^1 (1-r)^{\frac{1}{p}-2}M_1(f,r)dr < \varepsilon \cdots (2.1).$$

Since $M_1(f,r)$ is an increasing function of r, (2.1) remains valid when f is replaced by f_ρ. Now choose ρ so close to 1 that $|f_\rho(z) - f(z)| < \varepsilon$ on $|z| \le R$. Then

$$\int_0^R (1-r)^{\frac{1}{p}-2}M_1(f_\rho - f, r)dr < \varepsilon||1||_B.$$

Combining this with (2.1), we have

$$||f_\rho - f||_B \le \varepsilon||1||_B + 2\varepsilon,$$

so $f_\rho \to f$ in norm as $\rho \to 1$.

Lemma 2.11. *H^p is a dense subset of B^p.*

Lemma 2.12. *For each $f\varepsilon H^p$, $||f||_B \le c_p||f||_p$.*

Proof of Lemma 2.11, Lemma 2.12. Theorem 2.6 says that $H^p \subset B^p$, and gives the norm inequality. Also, H^p contains all functions holomorphic in a bigger disc, and such functions are dense in B^p by Lemma 2.10.

If we use above statements, the following fact is easily satisfied.

Proposition 2.13. *The space B^p with the given norm is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence. Since B^p lies in the L^1-space formed with respect to the measure

$$\iint \frac{1}{2\pi}(1-r)^{-2+\frac{1}{p}} dr d\theta,$$

the sequence $\{f_n\}$ convergences in the mean to a function $f \varepsilon L^1$. But this implies that some subsequences convergence pointwise almost everywhere to f. On the other hand, from Lemma 2.9 we see that $\{f_n\}$ converges uniformly on compact subsets, hence the limit function f is holomorphic in D. Thus $f \varepsilon B^p$.

Using all of the preceding properties, we can extend Theorem 2.8 as following.

Theorem 2.14. *B^p and H^p have the same continuous linear functionals; more precisely, Theorem 2.8 remains true if in its statements H^p is everywhere replaced by B^p.*

Proof. Let $\psi \varepsilon (B^p)^*$ be given and define the associated function $g(z) = \sum b_k z^k$ as in the proof of Theorem 2.8. By Lemma 2.12, ψ is also a bounded linear functional on H^p and hence g has the desired smoothness. Furthermore, if $f(z) = \sum a_k z^k \varepsilon B^p$, then by Theorem 2.8 we have

$$\begin{aligned} \psi(f) &= \lim_{\rho \to 1} \sum a_k z^k \rho^k \\ &= \lim_{\rho \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(\rho e^{i\theta}) g(e^{-i\theta}) d\theta \cdots (2.2) \end{aligned}$$

where $f_\rho \to f$ in norm, by Lemma 2.10.

Conversely let g (holomorphic and continuous) be given and suppose that g has the smoothness described in Theorem 2.8. We must show that the first limit in (2.2) exists for every $f \varepsilon B^p$ and is bounded by $c\|f\|$. The proof is identical to the proof of Theorem 2.8, using Theorem 2.5, Theorem 2.6, and Theorem 2.7.

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