Pusan Kyŏngnam Math. J. 9(1993), No. 1, pp 1–6

AN IDEAL CHARACTERIZATION OF COMMUTATIVE BCI-ALGEBRAS

JIE MENG

In our joint paper [6], the author of this paper and X. L. Xin introduced the concept of commutative BCI-algebras as a generalization of one of commutative BCK-algebras and a number of important properties of it were obtained. C. S. Hoo [2] also introduced the same concept called pseudo-commutative BCI-algebras. In my note [5], I introduced the concept of commutative ideals in BCK-algebras, by means of which some characterizations of commutative BCK-algebras were obtained. Now the results in [5] will be generalized to BCI-algebras. We define ~~commutative ideals and show that a BCI-algebra~~ is commutative if and only if every closed ideal of it is commutative. Moreover, we obtain an interesting results that let I be a commutative ideal and let A be a closed ideal, if $I \subset A$ then A is also a commutative ideal.

Let us recall some definitions and results which are necessary for development of this paper.

By a BCI-algebra is meant a set X with a binary operation $*$ and a constant 0 on it satisfying the following axioms:

(I) $(x * y) * (x * z) \leq z * y$,

(II) $x * (x * y) \leq y$,

(III) $x \leq x$,

(IV) $x \leq y$ and $y \leq x$ imply $x = y$,

(V) $x \leq y$ if and only if $x * y = 0$.

A BCI-algebra satisfying

(VI) $0 \leq x$

is called a BCK-algebra.

For any BCI-algebra X, the following hold:

(1) $x \leq 0$ implies $x = 0$,

(2) $x * 0 = x$,

(3) $(x * y) * z = (x * z) * y$,

(4) $x * (x * (x * y)) = x * y$,

Received January 15, 1993 .

(5) $0 * (x * y) = (0 * x) * (0 * y)$,

(6) $(x * z) * (y * z) \leq x * y$,

(7) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$.

A nonempty subset I of X is said to be an ideal if it satisfies:

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

We gave an equivalent condition of ideals which is needed for this paper.

PROPOSITION 1 ([4]). *A nonempty subset I of X is an ideal if and only if it satisfies (i) and*

(iii) *$y, z \in I$ and $x * y \leq z$ imply $x \in I$ for all x, y, z in X.*

Any ideal I have the following property:

(iv) $x \in I$ and $y \leq x$ imply $y \in I$.

For more details of BCI-algebras and its ideals we refer the readers to the references [1] and [2] listed in the end of this paper.

In this paper, unless otherwise specified, X will always mean a BCI-algebra.

DEFINITION 1. A nonempty subset I of X is called a commutative ideal if it satisfies (i) and

(v) $(x*y)*z \in I$ and $z \in I$ imply $x*((y*(y*x))*(0*(0*(x*y)))) \in I$ for all x, y, z in X.

Obviously, every ideal of a p-semisimple BCI-algebra is commutative([3]).

THEOREM 2. *Any commutative ideal must be an ideal.*

Proof. Let I be a commutative ideal of X and let $x * z \in I$ and $z \in I$, then $(x * 0) * z \in I$ and $z \in I$. From (v) it follows that $x = x * ((0 * (0 * x)) * (0 * (0 * (x * 0)))) \in I$, so I an ideal.

REMARK. An ideal may not be commutative. For instance, $X = \{0, 1, 2, 3, 4\}$, the operation $*$ is given by the following table:

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Then $(X;*,0)$ is a BCI-algebra, $\{0,1\}$ is an ideal of X but not commutative.

THEOREM 3. *An ideal I is commutative if and only if it satisfies*
(vi) $x*y \in I$ *implies* $x*((y*(y*x))*(0*(0*(x*y)))) \in I$.

Proof. This is routine and omitted.

DEFINITION 2 ([1]). An ideal I is closed if $0*x \in I$ whenever $x \in I$ for all x in X.

For a closed ideal the condition (vi) have a simpler form.

THEOREM 4. *Let I be a closed ideal of X, then I is commutative if and only if it satisfies*
(vii) $x*y \in I$ *implies* $x*(y*(y*x)) \in I$.

Proof. Suppose I is commutative and let $x*y \in I$, then $0*(x*y) \in I$ as I is closed and by (vi) we have $x*((y*(y*x))*(0*(0*(x*y)))) \in I$. Since

$$\begin{aligned}
&(x*(y*(y*x)))*(x*((y*(y*x))*(0*(0*(x*y)))))\\
&\le ((y*(y*x))*(0*(0*(x*y))))*(y*(y*x))\\
&= 0*(0*(0*(x*y)))\\
&= 0*(x*y) \in I,
\end{aligned}$$

it follows from Proposition 1 that $x*(y*(y*x)) \in I$, that is, I satisfies (vii).

Conversely, if I satisfies the condition (vii) and $x*y \in I$ then $x*(y*(y*x)) \in I$. Note that $0*(0*(x*y)) \in I$ by (iv). Since

$$\begin{aligned}
&(x*((y*(y*x))*(0*(0*(x*y)))))*(x*(y*(y*x)))\\
&\le (y*(y*x))*((y*(y*x))*(0*(0*(x*y))))\\
&\le 0*(0*(x*y)) \in I,
\end{aligned}$$

by Proposition 1 we have $x*((y*(y*x))*(0*(0*(x*y)))) \in I$, so I is commutative. The proof is complete.

With respect to the distribution of commutative ideals we have

THEOREM 5. *Given two ideals I and A of X with $I \subset A$, if A is closed and if I is commutative then A is also commutative.*

Proof. Assume that $x*y \in A$. For simplicity of notation let us write $u = x*y$, thus $0*u \in A$ as A is closed. Since I is commutative and $(x*u)*y = 0 \in I$, from (vi) it follows

$$(x*u)*(y*(y*(x*u))) = (x*y)*((y*(y*(x*u)))*(0*(0*((x*u)*y)))) \in I.$$

Note $I \subset A$, we have

$$(x*u)*(y*(y*(x*u))) \in A,$$

i.e.,

$$(x*(y*(y*(x*u))))*u \in A.$$

Combining $u \in A$ we obtain $x*(y*(y*(x*u))) \in A$. As

$$\begin{aligned}
&(x*(y*(y*x)))*(x*(y*(y*(x*u)))) \\
&\leq (y*(y*(x*u)))*(y*(y*x)) \\
&\leq (y*x)*(y*(x*u)) \\
&\leq (x*u)*x \\
&= 0*u \in A,
\end{aligned}$$

using Proposition 1 we have $x*(y*(y*x)) \in A$. This says that A is commutative. This finishes the proof.

DEFINITION 3 ([6]). A BCI-algebra X is called commutative if, for all x, y in X,

(8) $x \leq y$ implies $x = y*(y*x)$.

PROPOSITION 6 ([6]). *A BCI-algebra X is commutative if and only if it satisfies*

(9) $x*(x*y) = y*(y*(x*(x*y)))$.

Next we are ready to prove an important result.

THEOREM 7. *For any BCI-algebra X, the following are equivalent:*
(10) *X is commutative,*
(11) *every closed ideal of X is commutative,*
(12) *the zero ideal $\{0\}$ is commutative.*

Proof. (10) $\Rightarrow$ (11) Suppose X is commutative and I is a closed ideal of X. If $x * y \in I$ then $0 * (x * y) \in I$. By means of Proposition 6 we have

$$\begin{aligned}(x * (y * (y * x))) * (x * y) &= (x * (x * y)) * (y * (y * x)) \\ &= (y * (y * (x * (x * y)))) * (y * (y * x)) \\ &= (y * (y * (y * x))) * (y * (x * (x * y))) \\ &= (y * x) * (y * (x * (x * y))) \\ &\leq (x * (x * y)) * x \\ &= 0 * (x * y) \in I.\end{aligned}$$

Using Proposition 1 we get $x * (y * (y * y)) \in I$. This says that I is commutative. (11) holds.

(11) $\Rightarrow$ (12) It is immediate since $\{0\}$ is a closed ideal

(12) $\Rightarrow$ (10) If $x \leq y$ then $x * y = 0 \in \{0\}$. In virtue of Theorem 4 we know that $x * (y * (y * x)) \in \{0\}$, that is, $x * (y * (y * x)) = 0$. On the other hand, by (II) we have $(y * (y * x)) * x = 0$, hence $x = y * (y * x)$. This shows that X is commutative. (10) holds. The proof is complete.

Finally we discuss a quotient algebra of a BCI-algebra via a commutative ideal.

Suppose X is a BCI-algebra and I an ideal of X. An equivalent relation $\sim$ on X is defined by putting $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$, we denote the equivalent class containing x by C_x and let $X/I = \{C_x : x \in X\}$, then $(X/I; *, C_0)$ is a BCI-algebra, where $C_x * C_y = C_{x*y}$ for all x, y in X. If I is also closed then $C_0 = I$; otherwise $C_0 \neq I$.

THEOREM 8. *Let I be a closed ideal of X, then I is commutative if and only if $(X/I; *, C_0)$ is a commutative BCI-algebra.*

Proof. Suppose I is a closed commutative ideal of X. If $C_x * C_y = C_0$ then $x * y \in I$. By Theorem 4 we obtain $x * (y * (y * x)) \in I$, hence

$C_x * (C_y * (C_y * C_x)) = C_{x*(y*(y*x))} = I = C_0 \in \{C_0\}$. This says that the zero ideal $\{C_0\}$ is commutative for X/I. Therefore by Theorem 7, $(X/I; *, C_0)$ is commutative.

Conversely, let X/I be commutative, then by means of Theorem 7, $\{C_0\}$ is a commutative ideal of X/I. If $x * y \in I$ then $C_x * C_y = C_{x*y} = I = C_0 \in \{C_0\}$, and so $C_{x*(y*(y*x))} = C_x * (C_y * (C_y * C_x)) \in \{C_0\}$. This says that $x * (y * (y * x)) \in C_0 = I$. From Theorem 4 it sufficies to show that I is a commutative ideal of X. This completes the proof.

LEMMA 9. *Suppose* $(X; *, 0)$ *and* $(X'; *', 0')$ *are two BCI-algebras and let* $f : X \to X'$ *be a homomorphism, then* $Ker(f)$ *is a closed ideal of* X.

Proof. We have know that $Ker(f)$ is an ideal of X. In order to prove that it is closed, assume $x \in Ker(f)$ then $f(x) = 0'$. Since $f(0 * x) = f(0) * f(x) = 0' * 0' = 0'$, it follows that $0 * x \in Ker(f)$, namely $Ker(f)$ is a closed ideal of X, proving this lemma.

COROLLARY 10. *Let* $f : X \to X'$ *be an epimorphism, then* $Ker(f)$ *is a commutative ideal of* X *if and only if* X' *is a commutative BCI-algebra.*

Proof. The quotient algebra $X/Ker(f)$ is isomorphic to X' and use Theorem 8 and Lemma 9.

References

1. C. S Hoo, *Closed ideals and p-semisimple BCI-algebras*, Math. Japon **35** (1990), 1103–1112.
2. C. S Hoo, *Filters and ideals in BCI-algebras*, Math Japon. **36** (1991), 987–997
3. T. D. Lei and C C. Xi, *p-radical in BCI-algebras*, Math Japon. **30** (1985), 511–517.
4. J. Meng, *Ideals in BCK-algebras*, Pure appl. Math (in China) **2** (1986), 68–76, MR896:06009.
5. J. Meng, *Commutative ideals in BCK-algebras*, Pure Appl. Math (in China) **9** (1991), 49–53.
6. J. Meng and X. L. Xin, *Commutative BCI-algebras*, Math. Japon. **37** (1992), 569–572.

Department of Mathematics
Northwest University
Xian 710069, P. R. China