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NOTES ON A CLASS OF WEAKLY COMMUTATIVE SEMIGROUPS

S. Lajos

A semigroup S is called $(3,4)$-commutative if the identity $(x_1x_2x_3)(x_4x_5 x_6x_7) = (x_4x_5x_6x_7)\ (x_1x_2x_3)$ holds in S. The author shows that every $(3,4)$-commutative semigroup S is a semilattice of $(3,4)$-commutative archimedean [t-archimedean] semigroups; moreover S is a disjoint union of $(3,4)$-commutative power joined semigroups. A finitely generated periodic $(3,4)$-commutative semigroup is finite. A regular $(3,4)$-commutative semigroup is a semilattice of groups. Some results will be obtained concerning (m,n)-ideals of $(3,4)$-commutative semigroups.

Let S be a semigroup. We say that S is $(3,4)$-*commutative* if the identity

(1) $$(x_1x_2x_3)(x_4x_5x_6x_7) = (x_4x_5x_6x_7)(x_1x_2x_3)$$

holds for all elements $x_1, x_2, x_3, x_4, x_5x_6, x_7$ in S. It is easy to give examples of non-commutative but $(3,4)$-commutative semigroups of finite order. The following is an example of this kind:

	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	0	0	0	0	0
2	0	0	0	0	0	0
3	0	0	0	0	0	1
4	0	0	0	0	1	2
5	0	0	1	0	3	0.

In this paper we investigate some basic properties of $(3,4)$-commutative semigroups. For the terminology the reader is referred to [2] and [10], for (m,n)-ideals, see the author [5].

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Lemma 1. *Every $(3,4)$-commutative semigroup S is an $E-4$ semigroup.*
Proof. Let S be a $(3,4)$-commutative semigroup, $a, b \in S$. Then we have

$$\begin{aligned}(ab)^4 &= a(bab)abab = (a^2b)ab^2ab = (ab^2)a^3b^2 = (a^3b)bab^2 \\ &= (b^2a)b^2a^3 = (b^2a^3)b^2a = (b^2a^2)ab^2a = ab^2(b^2a^2)a \\ &= a^4b^4.\end{aligned}$$

Therefore $(ab)^4 = a^4b^4$ for every couple a, b of elements in S, that is, S is an $E-4$ semigroup, indeed.

Lemma 2. *Every $(3,4)$-commutative semigroup S is weakly commutative.*
Proof. This follows from Lemma 1, because of

$$(ab)^4 = a^4b^4 = b^4a^4, \tag{2}$$

and hence $(ab)^4 \in bSa$ for every pair $a, b \in S$.

Theorem 1. *Every $(3,4)$-commutative semigroup S is a semilattice of $(3,4)$-commutative archimedean [t-archimedean] semigroups.*

Proof. The statement follows from our Lemma 2 and II.5.6 Corollary in Petrich [10]. The bracketed case is an easy consequence of Lemma 2 above and a criterion due to Galbiati and Veronesi [3].

Theorem 2. *Every $(3,4)$-commutative semigroup S is a disjoint union of $(3,4)$-commutative power joined semigroups.*

Proof. The statement follows at once from our Lemma 1 and from Theorem 1.7 in [1].

Lemma 3. *A semigroup S is $(3,4)$-commutative if and only if the multiplicative semigroup $\mathbf{P}(S)$ of all nonempty subsets of S is $(3,4)$-commutative.*
Proof. In a $(3,4)$-commutative semigroup S the identity (1) holds. Thus the $(3,4)$-commutativity of $\mathbf{P}(s)$ follows from $(3,4)$-commutativity of S and from the definition of set product. Conversely, $(3,4)$-commutativity of $\mathbf{P}(S)$ implies that of S because $\mathbf{P}(s)$ contains all the one-element subsets of S.

Theorem 3. *Every $(3,4)$-commutative semigroup S is fixed point free 7-permutable.*

Proof. In a $(3,4)$-commutative semigroup S the identity (1) holds, and the permutation 4567123 is a fixed point free permutation of the set $\{1,2,3,4,5,6,7\}$.

Corollary 1. *A* $(3,4)$*-commutative semigroup* S *is fixed point free* n*-permutable for every positive integer* $n > 6$.

Proof. The identity (1) implies the identity

$$x_1x_2\cdots x_n = x_{n-3}x_{n-2}x_{n-1}x_nx_1x_2x_3\cdots x_{n-5}x_{n-4}, \tag{3}$$

and the cyclic permutation $(n-3, n-2, n-1, n, 1, 2, 3, \cdots, n-5, n-4)$ of the set $\{1, 2, \cdots, n\}$ is fixed point free for every integer $n > 6$.

Theorem 4. *Let* S *be a finitely generated periodic* $(3,4)$*-commutative semigroup. Then* S *is a finite semigroup.*

Proof. The statement of this theorem is a direct consequence of Theorem 3 and a well known result of Restivo and Reutenauer [11].

Theorem 5. *Let* S *be a intra-regular* $(3,4)$*-commutative semigroup. Then* S *is a semilattice of groups.*

Proof. For every element a of an intra-regular semigroup S there exist elements x, y in S such that

$$a = xa^2y. \tag{4}$$

Hence it follows that

$$\begin{aligned} a &= x(xa^2y)(xa^2y)y = (x^2a)ay(xa^2y)y \\ &= ay(xa^2y)y(x^2a), \end{aligned}$$

that is, $a \in aSa$ holds for every element a of S. Thus S is a regular semigroup.

On the other hand, we have

$$ex = exe = xe \tag{5}$$

for every idempotent element e and every element x of S. Therefore the set $E(S)$ of all idempotent elements of S is contained in the center of S, and hence S is a semilattice of groups, indeed.

The statement of Theorem 5 remains true with [left, right, quasi-. and completely] regular instead of "intra-regular". Hence we have the following result.

Theorem 6. *For a* $(3,4)$*-commutative semigroup* S *the following conditions are equivalent:*

(a) *S is regular.*
(b) *S is right regular.*
(c) *S is completely regular.*
(d) *S is quasi-regular.*
(e) *S is an inverse semigroup.*
(f) *S is a semilattice of rectangular groups.*
(g) *S is a commutative Clifford semigroup.*

Theorem 7. *If A is a globally idempoten bi-ideal of a* $(3,4)$*-commutative semigroup S, then A is a two-sided ideal of S.*

Proof. In this situation we obtain by Lemma 3,

$$ASA = AS = SA \subseteq A, \tag{6}$$

whence A is a two-sided ideal of S.

Theorem 8. *If A is a complete bi-ideal of a* $(3,4)$*-commutative semigroup S, then A is a two-sided ideal of S.*

Proof. By using our Lemma 3, we have

$$\begin{aligned} A &= ASA = (AS)(AS)ASA = ASA(AS)(AS) \\ &= SAS(ASA)A, \end{aligned}$$

whence it follows that A is a two-sided ideal of S.

Theorem 9. *Let S be a* $(3,4)$*-commutative semigroup and B a bi-ideal of S having the property* $B^2 = B^3$. *Then* B^2 *is a two-sided ideal of S.*

Proof. By using our Lemma 3, we get

$$SB^2 = SB^6 = B^4SB^2 \subseteq B^2SB \subseteq B^2,$$

and similarly,

$$B^2S = B^6S = B^3SB^3 \subseteq BSB^2 \subseteq B^2,$$

whence it follows that the power B^2 is a two-sided ideal of S.

Remark. For instance, the class of all regular semigroups does have the property $B^3 = B^2$ for every bi-ideal B.

Theorem 10. *Let S be a regular semigroup having a* $(3,4)$*-commutative bi-ideal semigroup* $\mathbf{B}(S)$. *Then S is a Clifford semigroup and* $\mathbf{B}(S)$ *is a commutative band.*

Proof. If S is a regular semigroup, then every bi-ideal B of S is complete, i.e.

$$B = BSB.$$

Hence $B = (BSB)(SB)(SB) = (SB)^2(BSB) = B(BSB)SBS$, whence it follows that every bi-ideal B of S is a two-sided ideal of S. Thus S is a regular duo semigroup which is a Clifford semigroup. Therefore, by a criterion of the author [7], the bi-ideal semigroup $\mathbf{B}(S)$ is a commutative band.

Theorem 11. *Let S be a π-regular $(3,4)$-commutative semigroup. Then S is a semilattice of nil-extensions of groups.*

Proof. This is a consequence of our Lemma 2 and a well known result due to S. Bogdanović.

Theorem 12. *Suppose that S is a $(3,4)$-commutative semigroup and A is an arbitrary $(3,4)$-ideal of S. Then A is a 7-ideal of S.*

Proof. By using the $(3,4)$-commutativity of the power semigroup $\mathbf{P}(S)$ we obtain

$$\begin{aligned} A^3SA^4 &= SA^7 = A^7S = A^5SA^2 = A(SA^2)A^4 = A^4SA^3 \\ &= A^6SA = A^2(SA)A^4 \subseteq A, \end{aligned}$$

whence A is a 7-ideal of S, indeed.

Theorem 13. *Suppose that S is a $(3,4)$-commutative semigroup and A is a complete $(0,k)$-ideal of S [resp. $(k,0)$-ideal of S], where k is an arbitrary fixed positive integer. Then A is a two-sided ideal of S.*

Proof. (i) $k = 1$. In this case we have

$$A = SA = S^6A = S^3AS^3,$$

whence A is a two-sided ideal of S.

(ii) $k = 2$. We have $A = SA^2 = (SA)^2(SA)^2 = (ASA^2)(SAS)$, and thus A is a two-sided ideal of S.

(iii) $k \geq 3$. In this case we have

$$A = SA^k = (SA^{k-1})(SA^k) = A^k(SA^{k-1}S),$$

whence it follows that A is a two-sided ideal of S. The proof is similar in the bracketed case, too.

Theorem 14. *Suppose that S is a $(3,4)$-commutative semigroup and A is a complete (m,n)-ideal of S, where m,n are fixed positive integers such that $m+n \geq 3$. Then A is a two-sided ideal of S.*

Proof. The proof of this theorem is very similar to that of Theorem 13, and thus we omit it.

Theorem 15. *Suppose that S is a $(3,4)$-commutative semigroup and A is a globally idempotent (m,n)-ideal of S, where m,n are arbitrary fixed non-negative integers such that $m+n \geq 1$. Then A is a two-sided ideal.*

Proof. We have

$$A^m S A^n = ASA = AS = SA = A$$

if m,n are positive integers. Hence A is a two-sided ideal of S. If $m=0$, $n>0$ we have

$$SA^n = SA^6 = A^3 S A^3 = A^3(A^3 S) = SA = AS \subseteq A,$$

whence A is a two-sided ideal of S. The proof is similar if $m>0$, $n=0$.

Theorem 16. *Let S be a $(3,4)$-commutative semigroup. Then the product $SE(S)$ is contained in the center of S.*

Proof. If x is an arbitrary element of a $(3,4)$-commutative semigroup S and $e \in E(S)$, then, for every $a \in S$, we have

$$a(xe) = (ea)xe = (xe)(ea) = (xe)a,$$

that is, every product xe is contained in the center of S.

Theorem 17. *Suppose that S is a $(3,4)$-commutative semigroup and A is an arbitrary (m,n)-ideal of S, where $m+n \geq 7$. Then A is an $(m+n)$-ideal of S.*

Proof. Proof is similar to that of Theorem 12 by using $(3,4)$-commutativity of the power semigroup $\mathbf{P}(S)$.

References

[1] Cherubini Spoletini, A. and A. Varisco, *Some properties of $E-m$ semigroups*, Semigroup Forum, 17(1979), 153-161.

[2] Clifford, A.H. and G.B. Preston, *The algebraic theory of semigroups I-II*, Amer. Math. Soc., Providence, R.I. 1964, 1967.

[3] Galbiati, J.L. and M.L. Veronesi, *Sui semigruppi che sono un band di t-semigruppi*, Ist. Lombardo, Rend. Sci., A114(1980), 217-234.

[4] Howie, J.M., *An introduction to semigroup theory*, Academic Press, London-New York-San Francisco, 1976.

[5] Lajos, S., *Generalized ideals in semigroups*, Acta Sci. Math., 22(1961), 217-222.

[6] Lajos, S., *On semilattices of groups*, Proc. Japan Acad., 45(1969), 383-384.

[7] Lajos, S., *A note on semilattices of groups*, Acta Sci. Math., 33(1972), 315-317.

[8] Lajos, S., *Characterizations of semigroups by (m,n)-ideals*, In "Notes on Semigroups VII", K. Marx Univ. Economics, Dept. Math., Budapest, No.4(1981), 25-29.

[9] Lajos, S., *Fibonacci characterizations and (m,n)-commutativity in semigroup theory*, Pure Math. Appl., Ser A, 1(1990), 59-65.

[10] Petrich, M., *Introduction to semigroups*, Merrill Books, Columbus, Ohio, 1973.

[11] Restivo, A. and C. Reutenauer, *On the Burnside problem for semigroups*, J. Algebra, 89(1984), 102-104.

Department of Mathematics, Budapest University for Economics, Kinizsi u. 1-7, H-1828 Budapest, Hungary