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SOME COMMUTATIVITY THEOREMS ON CERTAIN RINGS

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We prove (1) a semi-prime ring R satisfying the condition $[xyz, [xy, yx]] = 0$, is commutative provided $Char R \neq 2$ and also (2) a non-associative 2-torsion free ring with unity 1 satisfying the condition $[x^2y^2 - xy, x] = 0$ or $[x^2y^2 - xy, y] = 0$, is commutative.

Introduction

Initially Gupta [3] proved that division ring D which satisfies the polynomial identity $xy^2x = yx^2y$ for all $x, y \in D$ must be commutative. This was generalized by Awtar [2] as "A semi-prime ring R satisfying the condition $xy^2x - yx^2y \in Z(R)$ is commutative." Al-Mojil generalized this theorem by showing that a 2-torsion free semi-prime ring in which xy commutes with $xy^2x - yx^2y$ for all $x, y \in R$, is commutative.

We generalize this result as "In a 2-torsion free semi-prime ring, if xyz commutes with $xy^2x - yx^2y$ for all $x, y, z \in R$, then R is commutative.

A result proved by Ashraf and Quadri [5] is that a semi-prime ring satisfying the condition $(xy)^2 - xy \in Z(R)$ is commutative. We generalize this result by showing that a non-associative ring satisfying either of the conditions $[x^2y^2 - xy, x] = 0$ and $[x^2y^2 - xy, y] = 0$, is commutative provided $Char R \neq 2$.

Lemma 1.1. *Posner [6, Theorem 1] : Let R be a 2-torsion free (namely $Char R \neq 2$) prime-ring and d_1, d_2 derivations of R such that the iterate d_1, d_2 is also a derivation. Then one at least of d_1, d_2 is zero.*

Lemma 1.2. *Herstein [4, Theorem]: Let R be a ring having no non-zero nil ideals in which for every $x, y \in R$ there exists integer $m = m(x, y) \geq 1$, $n = n(x, y) \geq 1$ such that $[x^m, y^n] = 0$. Then R is commutative.*

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Theorem 1.3. *Let R be a 2-torsion free semi-prime ring such that $[xyz,[xy,yx]]=0$ for all x,y in R. Then R is commutative.*

Proof. By hypothesis,

$$(xy^2x-yx^2y)xyz=xyz(xy^2x-yx^2y) \tag{1}$$

Replacing x by $x+y$ in (1), we obtain

$$\begin{aligned}&(xy^2x-yx^2y)(xyz+y^2z)+[y^2(yx-xy)+(xy-yx)y^2](xyz+y^2z)\\&=(xyz+y^2z)(xy^2x-yx^2y)+(xyz+y^2z)\\&\quad[y^2(yx-xy)+(xy-yx)y^2]\end{aligned} \tag{2}$$

Again replace x by $x+y$ in (2) to yield

$$\begin{aligned}&[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2][xyz+y^2z+y^2z]\\&\quad+[y^2(yx-xy)+(xy-yx)y^2][xyz+y^2z+y^2z]\\&=(xyz+y^2z+y^2z)[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2]\\&\quad+(xyz+y^2z+y^2z)[y^2(yx-xy)+(xy-yx)y^2]\end{aligned} \tag{3}$$

Using (2) in (3), we obtain

$$\begin{aligned}&[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2]y^2z\\&\quad+[y^2(yx-xy)+(xy-yx)y^2][xyz+y^2z+y^2z]\\&=y^2z[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2]\\&\quad+[xyz+y^2z+y^2z][y^2(yx-xy)+(xy-yx)y^2]\end{aligned} \tag{4}$$

Again replacement of x by $x+y$ in (4) yields

$$\begin{aligned}&[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2]y^2z\\&\quad+[y^2(yx-xy)+(xy-yx)y^2]y^2z\\&\quad+[y^2(yx-xy)+(xy-yx)y^2][xyz+y^2z+y^2z+y^2z]\\&=y^2z[xy^2x-yx^2y+y^2(yx-xy)+(xy-yx)y^2]\\&\quad+[xyz+y^2z+y^2z+y^2z][y^2(yx-xy)+(xy-yx)y^2]\end{aligned}$$

Simplifying and using (4), we obtain

$$2[y^2(yx-xy)+(xy-yx)y^2]y^2z=2y^2z[y^2(yx-xy)+(xy-yx)y^2].$$

But since R is 2-torsion free, so we obtain

$$[y^2(yx-xy)+(xy-yx)y^2]y^2z=y^2z[y^2(yx-xy)+(xy-yx)y^2].$$

i.e.

$$[(xy-yx)y^2-y^2(xy-yx)]y^2z=y^2z[(xy-yx)y^2-y^2(xy-yx)] \quad (5)$$

Replacing z by $z+y$ in (5) and using (5) we obtain

$$[(xy-yx)y^2-y^2(xy-yx)]y^3z=y^3[(xy-yx)y^2-y^2(xy-yx)]$$

i.e.

$$[[[x,y],y^2],y^3]=0 \quad (6)$$

Let I_r denote the inner derivation with respect to r i.e. $I_r : X \to [r,x]$, then (6) becomes $I_{y^3}I_{y^2}I_y(x)=0$. Using lemma 1.1 which is applicable in prime rings we have either $I_{y^3}I_{y^2}=0$ or $I_y=0$. If $I_{y^3}I_{y^2}=0$ then $I_{y^3}I_{y^2}(x)=0$ for $x,y\in R$. Then again by lemma 1.1. Either $I_{y^3}=0$ or $I_{y^2}=0$ i.e. $y^3\in Z(R)$ or $y^2\in Z(R)$. Then in both the cases we have either $[y^3,x]=0$ or $[y^2,x]=0$ which by lemma 2.2 yields that R is commutative. Now consider the case $I_y=0$ which implies $I_y(x)=0$ or $xy-yx=0$ i.e. $xy=yx$.

Thus in all the cases R is commutative. Since R is isomorphic to subdirect sum of prime ring R_α each of which as homomorphic image of R satisfies the hypothesis imposed on R so theorem holds for semi-prime rings also.

Theorem 1.4. *A non-associative ring with unity 1 satisfying either of the conditions:*

$$\text{(a) } [x^2y^2-xy,x]=0 \qquad \text{(b) } [x^2y^2-xy,y]=0$$

is commutative provided it is 2-torsion free.

Proof. By hypothesis (a) we have

$$(x^2y^2-xy)x=x(x^2y^2-xy) \quad (7)$$

Replacing y by $y+1$ in (7) and using it, we obtain

$$2x(x^2y)=2(x^2y)x$$

Since R is 2-torsion free hence

$$x(x^2y)=(x^2y)x. \quad (8)$$

Now replacing x by $x+1$ in (8) and using it we yield

$$2x(xy) + xy = 2(xy)x + yx \tag{9}$$

Again replacing x by $x+1$ and using (9) we obtain

$$2xy = 2yx$$

i.e. $2(xy - yx) = 0$. But R is 2-torsion free.

So we have $xy = yx$. Thus R is commutative. Hypothesis (b) gives us

$$(x^2y^2 - xy)y = y(x^2y^2 - xy) \tag{10}$$

Replacing x by $x+1$ in (10) and using (10) we obtain

$$2(xy^2)y = 2y(xy^2)$$

But R is 2-torsion free, hence

$$(xy^2)y = y(xy^2)$$

Now replace y by $y+1$ in (11) and use (11) to obtain

$$2(xy)y + xy = 2y(xy) + yx \tag{11}$$

Again replacing y by $y+1$ in (12) and using (12) we obtain $2(xy-yx) = 0$, since R is 2-torsion free, this yields $xy = yx$. Thus R is commutative.

Example. The rings $R_1 = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in GF(2) \right\}$ and $R_2 = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \middle| a, b \in GF(2) \right\}$ respectively exhibit that polynomial identities $[x^2y^2 - xy, x] = 0$ and $[x^2y^2 - xy, y] = 0$ are satisfied though these rings are non commutative, without identity. This shows that in hypothesis of theorem 1.4 existence of identity and $Char R \neq 2$, are essential conditions.

References

[1] Abdullh-H-Almojil, *On commutativity of semi-prime rings*, J. Austral. Math. Soc. (Series A) 32-1982, 48-51.

[2] Awtar R., *A remark on commutativity of certain rings*, Proc. Amer. Math. Soc. 41(1973), 370-372.

[3] Gupta R.N., *Nilpotent matrices with invertible transpose*, Proc. Amer. Math. Soc. 24-1970, 372-575.

[4] Herstein I.N., *A commutativity theorem*, J. Algebra 38 (1976), 112-118.

[5] M.A. Quadri, Mohd. Ashraf and M.A. Khan, *A commutativity conditions for semi-primerings II*, Bull. Austral. Math. Soc. Vol:33(1986), 71-73.

[6] Posner E.C., *Derivation in prime rings*, Proc. Amer. Math. Soc. 8(1957), 1093-1100.

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