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ON SQUARES OF JACOBSON RADICLAS

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In this paper, we investigate the square of Jacobson radical of rings. We prove that the square of Jacobson radical is also a radical property.

Let R be a class of rings. We say that the ring A is an R-ring if A is in R. An ideal I of A will be called an R-ideal if I is an R-ring. A ring which does not contain any non-zero R-ideals will be called R-semisimple. We shall call R a radical property or radical class if the following three conditions hold:

(A) homomorphic image of an R-ring is an R-ring,

(B) every ring A contains a largest R-ideal S,

(C) the quotient ring A/S is R-semisimple.

The largest R-ideal S of a ring A is called the R-radical of A.

We need the following theorem.

Theorem 1. [1] [2] *R is a radical class if, and only if R satisfies conditions (A), (B), and (D).*

(D) If I is an R-ideal of A and A/I is a R-ring.

Now let $J(A)$ be the Jacobson radical of A and $J^2 = \{A|(J(A))^2 = A\}$.

Theorem 2. *J^2 is a radical property.*

Proof. If $A \in J^2$, then $A = J^2(A) = (J(A))^2$. If $f : A \to B$ is a homomorphism from A onto B, then

$$\begin{aligned} B &= f(A) = f((J(A))^2) = [f(J(A))]^2 \subseteq [J(f(A))]^2 \\ &= [J(B)]^2 = J^2(B). \end{aligned}$$

Therefore $B \in J^2$.

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Next we wish to show that every ring A contains a largest J^2-ideals S. Let $\{I_k | i \in N\}$ be the set of all J^2-ideals of A. Then $J(\sum\{I_i | i \in N\}) = \sum\{J(I_i) | i \in N\} = \sum\{I_i | i \in N\}$ and $[J(\sum\{I_i | i \in N\})]^2 = [\sum\{I_i | i \in N\}]^2 = \sum_{i \in N} I_i^2 + \sum\{I_i I_j \mid i, j \in N\} = \sum\{I_i | i \in N\}$. Hence $\sum\{I_i | i \in N\}$ is the largest J^2-ideal of A.

Now we assume that $(J(A/I))^2 = A/I$ and $(J(I))^2 = I$. We wish to show that $A = (J(A))^2$. Let $a \in A$, then $a = \sum_{i=1}^n b_i c_i + x$ for some b_i, c_i, x with $b_i + I \in J(A/I)$. $c_i + I \in J(A/I)$ and $x \in I$. Hence there exist d_i, e_i and

$$(d_i + I) \circ (b_i + I) = I, \quad (e_i + I) \circ (c_i + I) = I.$$

Hence $d_i \circ b_i = y_i \in I$, $e_i \circ c_i = z_i \in I$. $I = (J(I))^2 \subseteq J(I)$, there are d_i', e_i' in I such that

$$\begin{aligned} d_i' \circ (d_i \circ b_i) &= (d_i' \circ d_i) \circ b_i = 0 \\ e_i' \circ (e_i \circ c_i) &= (e_i' \circ e_i) \circ c_i = 0. \end{aligned}$$

Hence $b_i \in J(A), c_i \in J(A)$ for each i, and $x = \sum_{j=1}^m f_j g_j$ with $f_j \in I \subset J(A)$, $g_j \in J(A)$. We have $a = \sum_{i=1}^n b_i c_i + \sum_{j=1}^m f_j g_j \in (J(A))^2$. Therefore $A = (J(A))^2$ and $A \in J^2$. By Theorem 1, J^2 is a radical property.

References

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