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HAUSDORFF DIEMENSION OF CANTOR-LIKE SETS

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1. Introduction

Hausdorff showed that the Hausdorff dimension of Cantor ternary set is $\log 2/\log 3$. H.H. Lee and C.Y. Park [1] calculated the Hausdorff dimension of some symmetric Cantor sets. They used Lebesgue function to calculate them. In this paper, we extend the notion of Lebesgue function and we find the Hausdorff dimension of some 'Cantor-type' sets, which form a larger class than the class of symmetric Cantor sets considered by H.H. Lee and C.Y. Park.

2. Definitions

Suppose that F is a subset of R. For every positive α and ε, put

$$H_\alpha^\varepsilon(F) = \inf \sum_{n=1}^{\infty} [d(A_n)]^\alpha$$

where the infimum runs over all countable coverings of F by sets A_n with diameter, $d(A_n) = \sup\{|x-y| : x, y \in A_n\}$, less than ε. For every positive α, $H_\alpha(F) = \sup H_\alpha^\varepsilon(F)$ where the supremum runs over all positive $\varepsilon > 0$.

It is known that [2] there exists a unique point $\alpha_0 \in [0, \infty]$ such that $H_\alpha(F) = \infty$ for $\alpha < \alpha_0$ and $H_\alpha(F) = 0$ for $\alpha > \alpha_0$. This value α_0 is called the *Hausdorff dimension of* F and it is denoted by $\dim F$.

Let $(k_n)_{n=1}^\infty$ be a fixed sequence of positive integers such that $k_n \geq 2$, for every n. Let $(c_n)_{n=0}^\infty$ be another fixed sequence of positive numbers such that

$$0 < k_n c_n < c_{n-1}, \text{ for every } n = 1, 2, \cdots$$

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Define, for every natural number n,

$$r_n = \frac{c_{n-1} - c_n}{k_n - 1}.$$

Let $F_n = \{\sum_{j=1}^{n} \varepsilon_j r_j : \varepsilon_j = 0, 1, 2, \cdots k_j - 1, j = 1, 2, \cdots n\}$ and let $F_0 = \{0\}$. Define $E_n = \cup_{t \in F_n} [t, t + c_n]$. The set $E = \cap_{n=1}^{\infty} E_n$ will be called the *Cantor-like* set determined by the given two fixed sequences $(k_n)_{n=1}^{\infty}$ and $(c_n)_{n=0}^{\infty}$.

Let E be a Cantor-like set which is defined by two sequences $(k_n)_{n=1}^{\infty}$ and $(c_n)_{n=0}^{\infty}$. Let $h_n = (k_1, k_2, \cdots k_n)^{-1}$. The *Lebesgue function* L on $[0, c_0]$ is defined as follows: For $\sum_{i=1}^{\infty} \varepsilon_i r_i \in E$, let

$$L(\sum_{i=1}^{\infty} \varepsilon_i r_i) = \sum_{i=1}^{\infty} \varepsilon_i h_i.$$

For each $n \geq 0$, $t = \sum_{i=1}^{n} \varepsilon_i r_i \in F_n$, we have

$$\begin{aligned} L(t + c_{n+1}) &= L(t + \sum_{i=n+2}^{\infty} (k_i - 1) r_i) \\ &= \sum_{i=1}^{n} \varepsilon_i h_i + \sum_{i=n+2}^{\infty} (k_i - 1) h_i = \sum_{i=1}^{n} \varepsilon_i h_i + h_{n+1} \\ &= L(t + r_{n+1}). \end{aligned}$$

We define L on $(t + c_{n+1}, t + r_{n+1})$ to have the constant value that it has at both of the end points, namely, $L(t) + h_{n+1}$. Thus we have defined the function L on $[0, c_0]$. Note that L is a monotone nondecreasing and continuous on $[0, c_0]$ and L is constant on each component interval of $[0, c_0] - E$.

Define a function $w_L : (0, \infty) \to [0, \infty]$ by

$$w_L(t) = \sup\{|L(x_2) - L(x_1)| : [x_1, x_2] \subseteq [0, c_0] \text{ and } |x_2 - x_1| \leq t\}.$$

We follow these notations in the remaining part of this paper. For us, measure means an outer measure.

3. Results

We need the following Frostman's result what was used by Lee and Park. However we give the proof for completeness.

Theorem 1. *Let I be a closed interval in the real line. Suppose that there exists a positive measure μ on I such that*

$$\begin{aligned} f(t) &= \sup\{\mu([x_1, x_2]) : [x_1, x_2] \subseteq I, 0 < x_2 - x_1 \leq t\} \\ &= o(t^\alpha)(t \downarrow 0) \text{ for some } \alpha > 0. \end{aligned}$$

Then $H_\alpha(A)$ is positive (so $\dim A \geq \alpha$) whenever A is a compact subset of I with $\mu(A) > 0$.

Proof. Choose $0 < c < \infty$ such that $f(t) \leq ct^\alpha$ for $0 < t \leq \delta_0$, for some δ_0. Consider any open cover $\{(u_i, v_i)\}_{i=1}^\infty$ of A such that $v_i - u_i \leq \delta$, where $\delta \leq \delta_0$. Then we have

$$f(v_i - u_i) \leq c(v_i - u_i)^\alpha \text{ for every } i.$$

This implies

$$\begin{aligned} \sum_{i=1}^\infty (v_i - u_i)^\alpha &\geq \frac{1}{c}\sum_{i=1}^\infty f(v_i - u_i) \\ &\geq \frac{1}{c}\sum_{i=1}^\infty \mu([u_i - v_i]) \\ &\geq \frac{1}{c}\mu(A) \\ &> 0. \end{aligned}$$

Therefore, $H_\alpha^\delta(A) \geq \frac{1}{c}\mu(A)$ where $\delta \leq \delta_0$ and this implies that $H_\alpha(A) \geq \frac{1}{c}\mu(A)$. Thus $H_\alpha(A)$ is positive (so $\dim A \geq \alpha$).

Lemma 1. *For each natural number n,*

$$h_n \leq w_L(c_n) \leq 2h_n.$$

Proof. Since $L(c_n) - L(0) = h_n$, the first inequality is clear. Note that L is a continuous monotone nondecreasing function. Fix $x, y \in [0, c_0]$ with $x < y$ and $|x - y| \leq c_n$. If $x \in [c_0 - c_n, c_0]$ then

$$L(y) - L(x) \leq L(c_0) - L(c_0 - c_n) = h_n.$$

If $x \in [0, c_0 - c_n]$ then define

$$t_1 = \max\{t \in F_n : t \leq x\}$$

and

$$t_2 = \min\{t \in F_n : x < t\}.$$

If $x \in [0, c_0 - c_n]$ then

$$\begin{aligned} L(y) - L(x) &\leq L(t_2 + c_n) - L(t_1) \\ &= L(t_2 + c_n) - L(t_2) + L(t_2) - L(t_1) \\ &= h_n + h_n. \end{aligned}$$

Therefore the second inequality in the statement holds too.

Theorem 2. *Suppose that $t^\alpha = o(w_L(t))(t \downarrow 0)$ for some $\alpha > 0$. Then $H_\alpha(E)$ is finite (so $\dim E \leq \alpha$).*

Proof. Choose $0 < c < \infty$ such that $t^\alpha \leq c w_L(t)$ for $0 < t \leq c_0$. Since $E \subseteq E_j$ and E_j is the union of h_j^{-1} intervals of length c_j, and c_j tends to zero as j tends to infinity, by the previous lemma 1 we have

$$\begin{aligned} H_\alpha(E) &\leq \liminf_{n\to\infty} h_n^{-1} c_n^\alpha \\ &\leq c \liminf_{n\to\infty} h_n^{-1} w_L(c_n) \\ &\leq 2c. \end{aligned}$$

Theorem 3. *Let $\alpha = \liminf_{n\to\infty} \dfrac{\log h_n}{\log c_{n+1}}$ and $\beta = \limsup_{n\to\infty} \dfrac{\log h_{n+1}}{\log c_n}$. Then $\dim E \in [\alpha, \beta]$.*

Proof. Choose j with $c_j < 1$. For given t with $0 < t \leq c_j$, choose $n \geq j$ such that $c_{n+1} \leq t \leq c_n$. Then $-\log c_{n+!} \geq -\log t \geq -\log c_n > 0$. By the lemma 1, $-\log 2h_n \leq -\log w_L(c_n) \leq -\log h_n$, for all $n \geq 1$. Since w_L is nondecreasing, we also have $0 < -\log w_L(c_n) \leq -\log w_L(t) \leq -\log w_L(c_{n+1})$ for our t and $n \geq j$.

These three set of inequalities yield

$$\begin{aligned} 0 &\leq \frac{\log 2h_n}{\log c_{n+1}} \leq \frac{\log w_L(c_n)}{\log c_{n+1}} \leq \frac{\log w_L(c_n)}{\log t} \leq \frac{\log w_L(t)}{\log t} \\ &\leq \frac{\log w_L(c_{n+1})}{\log t} \leq \frac{\log h_{n+1}}{\log t} \leq \frac{\log h_{n+1}}{\log c_n}. \end{aligned}$$

In particular,

$$\frac{\log 2}{\log c_{n+1}} + \frac{\log h_n}{\log c_{n+1}} \leq \frac{\log w_L(t)}{\log t} \leq \frac{\log h_{n+1}}{\log c_n} \text{ and } 0 \leq \alpha \leq \beta.$$

Therefore, given $\varepsilon > 0$, there exists some $t_0 \in [0, c_0] \cap [0, 1)$ such that

$$\alpha - \varepsilon < \frac{\log w_L(t)}{\log t} < \beta + \varepsilon \text{ whenever } 0 < t < t_0 < 1.$$

That is, $t^{\alpha-\varepsilon} \geq w_L(t) \geq t^{\beta+\varepsilon}$ whenever $0 < t < t_0$. Now the theorem 1 applied to

$$\mu(A) = \inf\{\sum_{i=1}^{\infty}[L(y_i) - L(x_i)] : A \subseteq \cup_{i=1}^{\infty}[x_i, y_i] \subseteq [0, c_0]\},$$

when $A \subseteq [0, c_0]$, implies that $\dim E \geq \alpha - \varepsilon$ and theorem 2 implies that $\dim E \leq \beta + \varepsilon$. Therefore $\dim E \in [\alpha, \beta]$.

Lemma 2. *If there is a constant k such that $k_n \leq k$, for every n, then*

$$\begin{aligned} \liminf_{n\to\infty} \frac{\log h_n}{\log c_{n+1}} &= \liminf_{n\to\infty} \frac{\log h_n}{\log c_n} = \liminf_{n\to\infty} \frac{\log h_n}{\log r_n} \text{ and} \\ \limsup_{n\to\infty} \frac{\log h_{n+1}}{\log c_n} &= \limsup_{n\to\infty} \frac{\log h_n}{\log c_n} = \limsup_{n\to\infty} \frac{\log h_n}{\log r_n}. \end{aligned}$$

Proof. Notice that

$$\begin{aligned} \frac{\log h_n}{\log c_{n+1}} &= \frac{\log h_{n+1}}{\log c_{n+1}} + \frac{\log k_{n+1}}{\log c_{n+1}} \\ \frac{\log h_{n+1}}{\log c_n} &= \frac{\log h_n}{\log c_n} - \frac{\log k_{n+1}}{\log c_n}. \end{aligned}$$

Also $0 > \frac{1}{\log r_{n+1}} > \frac{1}{\log c_n} > \frac{1}{\log r_n}$ so that $0 < \frac{\log h_n}{\log r_{n+1}} < \frac{\log h_n}{\log c_n} < \frac{\log h_n}{\log r_n}$, when n is sufficiently large. That is

$$\frac{\log h_n}{\log r_n} > \frac{\log h_n}{\log c_n} > \frac{\log h_{n+1}}{\log r_{n+1}} + \frac{\log k_{n+1}}{\log r_{n+1}}.$$

The lemma follows from these relations, because

$$\lim_{n\to\infty} \frac{\log k_n}{\log c_n} = \lim_{n\to\infty} \frac{\log k_n}{\log r_n} = 0.$$

Corollary 1. *If* $\liminf_{n\to\infty} h_{n+1}^{-1} c_n^{\alpha}$ *is a finite positive constant for some* $\alpha \in (0, \infty)$, *then* $\liminf_{n\to\infty} \frac{\log h_{n+1}}{\log c_n} = \alpha = \dim E$.

Proof. Suppose that $\liminf_{n\to\infty} h_{n+1}^{-1} c_n^\alpha = k$ for some $k \in (0,\infty)$. Then

$$\liminf_{n\to\infty} \log(h_{n+1}^{-1} c_n^\alpha) = \log k < \infty$$

and so $\liminf_{n\to\infty} \dfrac{\log h_{n+1}}{\log c_n} - \alpha = \liminf_{n\to\infty} \dfrac{-\log h_{n+1}}{-\log c_n} - \alpha = 0.$

Now the theorem 3 implies that $\dim E \geq \alpha$. Also

$$\begin{aligned} H_\alpha(E) &\leq \liminf_{n\to\infty} h_n^{-1} c_n^\alpha \\ &\leq \liminf_{n\to\infty} h_{n+1}^{-1} c_n^\alpha < \infty \text{ so that } \alpha \geq \dim E. \end{aligned}$$

Therefore, $\dim E = \alpha = \liminf_{n\to\infty} \dfrac{\log h_{n+1}}{\log c_n}$.

In the same way we can also prove the following corollary.

Corollary 2. *If $\liminf_{n\to\infty} h_n^{-1} c_n^\alpha$ is a finite positive constant for some $\alpha \in (0,\infty)$ and $(k_n)_{n=1}^\infty$ is a bounded sequence then*

$$\liminf_{n\to\infty} \frac{\log h_n}{\log c_n} = \alpha = \dim E.$$

Corollary 3. *If $(k_n)_{n=1}^\infty$ is a bounded sequence, and $\lim_{n\to\infty} \frac{\log h_n}{\log c_n} = \alpha$ exists, then $\dim E = \alpha = \lim_{n\to\infty} \frac{\log h_n}{\log r_n}$.*

Corollary 4. *If $\lim_{n\to\infty} \frac{c_n}{c_{n-1}} = a, k_m = k$ for every m, for some constant $k \geq 2$ then*

$$\dim E = \frac{\log k}{-\log a}.$$

Proof. Notice that $\lim_{n\to\infty} (c_n)^{1/n} = a$ and so $\lim_{n\to\infty} \frac{\log h_n}{\log c_n} = \frac{\log k}{-\log a}$.

References

[1] H.H. Lee and C. Y. Park, *Hausdorff dimension of symmetric Cantor sets*, Kyungpook Math. J., 28, 1988, 141-146.

[2] C.A. Rogers, *Hausdorff Measures*, Cambridge University Press, 1970.

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