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A NOTE ON SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS

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In this paper we wish to study the boundedness and asymptotic behaviour of certain class of second order integro-differential equations of the form

$$(a(t)x')' + b(t)x' + c(t)x = K(t, x, \int_0^t H(s, x, x')ds) + Q(t, x, x')$$

by means of comparison with the solutions of the second order linear differential equation

$$(a(t)x')' + b(t)x' + c(t)x = 0.$$

1. Introduction

Recently Grace and Lalli [3], and Agarwal [1] studied the asymptotic behaviour of the integro-differential equation

$$(a(t)x')' + b(t)x' + c(t)x = r(t)\int_0^t g(s)x(s)ds + p(t, x, x') \cdots \tag{1.1}$$

as $t \to \infty$. It was proved that under some suitable conditions on the functions $r(t), g(t)$, and $p(t, x, x')$, there is a solution of (1.1) satisfying any given initial conditions which tends to a solution of the linear differential equation

$$(a(t)x')' + b(t)x' + c(t)x = 0 \tag{1.2}$$

for which the general solution is known. The purpose of this note is to extend earlier results of Grace and Lalli [3] for equation (1.1), and

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generalize previously known results of Mehri and Zarghame [4], Bellman [2, p. 112], Trench [7], Pachpatte [6]. Instead of equation (1.1) we shall investigate an equation of the form

$$(a(t)x')' + b(t)x' + c(t)x = K(t, x, \int_0^t H(s, x, x')ds) + Q(t, x, x'). \quad (1.3)$$

Our main assumptions are:

1. Let us assume that if $|x(t)| < p(t)u$ and $|x'(t)| \leq q(t)u$, then there are continuous functions F_1 and F_2, positive, monotonic nondecreasing and satisfying:

(a1) $F_1(0) = 0$, $F_1(u)$ is submultiplicative for $u > 0$,

(a2) $\Omega(r) = \int_{r_0}^{r}(1/F_1(s))ds \to \infty$ as $r \to \infty, r_0 > 0$.

(a3) $F_2(x)/n \leq F_2(\frac{x}{n}), n > 0$

(a4) $G(r) = \int_{r_0}^{r}(1/s + F_2(s))ds \to \infty$ as $r \to \infty, r_0 > 0$.

and such that the following conditions hold:

(b1) $|Q(t, x, x')| \leq C_1(t)u + C_2(t)F_1(u)$,

(b2) $|K(t, x, y)| \leq C_3(t)u^{\alpha}|y|, \alpha \in [0, 1]$

(b3) $|H(t, x, x')| \leq 2C_4(t)F_2(u)$

where $C_i(t), i = 1, 2, 3, 4$ are continuous, positive functions defined on I, $I = [0, \infty)$.

2. $p(t) \geq \max\{|z_1(t)|, |z_2(t)|\}, q(t) \geq \max\{|z_1(t)|, |z_2(t)|\}$ where z_1, z_2 are any two linearly independent solutions of (2).

In proofing of the results, the following lemma will be used.

Lemma A (Pachpatte [5]): *Let $x(t), g(t), h(t)$ be real-valued positive continuous functions defined on I, $H : I \to I$ such that $H(u)$ is positive nondecreasing and continuous for $u > 0$*

$$|H(u)/n| \leq H|u/n|, n > 0,$$

and $w(u)$ be positive, continuous, monotonic, nondecreasing and submultiplicative function for $u > 0$, $W(0) = 0$ and suppose further the inequality

$$x(t) \leq x_0 + \int_0^t g(s)x(s)ds + \int_0^t g(s)\int_0^s g(\tau)H(x(\tau))d\tau ds + \int_0^t h(s)W(x(s))ds$$

is satisfied for all $t \in I$, where x_0 is a positive constant. Then

$$\begin{aligned} x(t) \ \leq \ & \Omega^{-1}[\Omega(x_0) + \int_0^t h(s)W(1 + \int_0^s g(\tau)G^{-1}[C(\tau)]d\tau)ds] \\ & \cdot[1 + \int_0^t g(s)G^{-1}|C(s)|ds] \end{aligned}$$

where

$$C(t) = G(1) + \int_0^t g(s)ds, G(r) = \int_{r_0}^{r} [1/s + H(s)]ds, r \geq r_0 > 0$$

and G^{-1} *is the inverse of* G *and* $t \in [0, b] \in I$ *so that* $C(t) \in Dom(G^{-1})$.

$$\Omega(r) = \int_{r_0}^{r} [1/W(s)]ds, r \geq r_0 > 0$$

and Ω^{-1} *is the inverse of* Ω *and* $t \in [0, b]$ *such that* $C(t) \in Dom(g^{-1})$ *and*

$$(x_0) + \int_0^t h(s)W|1 + \int_0^s g(\tau)G^{-1}(C(\tau))d\tau|ds \in Dom(\Omega^{-1}).$$

2. Main Result

In this note we investigate the boundedness and the asymptotic behaviour of the integro-differential equation (1.3). We also prove that if the solutions of (1.2) are bounded, so are the solutions of (1.3) provided the functions K and Q are properly chosen.

Theorem 2.1. *In addition to the above assumption, let the following conditions hold:*

(i) $a(t), b(t) \& c(t)$ *are continuous functions for* $t \in I$, *and* $a(t) > 0$.

(ii) $r(t)$ *and* $p(t)C_3(t)/a(t)W(t)$ *are in* $L(0, \infty)$ *where* $W(t)$ *is the Wronskian of* z_1 *and* z_2 *and*

$$r(t) = \max\{\frac{p(t)C_1(t)}{a(t)W(t)}, \frac{p(t)C_3(t)}{a(t)W(t)}, C_4(t)\}. \tag{2.1}$$

Then for every pair (x_0, x_0') *of numbers, there is a solution of (1.3) which can be written in the form*

$$x(t) = A(t)z_1(t) + B(t)z_2(t) \tag{2.2}$$

satisfying the initial conditions $x(0) = x_0$ *and* $x'(0) = x_0'$, *with*

$$\lim_{t\to\infty} A(t) = \ell \text{ and } \lim_{t\to\infty} B(t) = m.$$

Proof. Let $x(t)$ be a solution of (1.3) and is written the form (2.2). Applying variation of parameter formula for finding solutions of equation (1.3) we obtain

$$\begin{aligned} A(t) &= A(0) + \int_0^t [z_2(s)h(s)/W(s)]ds, \\ B(t) &= B(0) - \int_0^t [z_1(s)h(s)/W(s)]ds \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} h(t) &= \frac{1}{a(t)}[K(t, Az_1 + Bz_2, \int_0^t H(s, Az_1 + Bz_2, Az_1' + Bz_2')ds)] \\ &+Q(t, Az_1 + Bz_2, Az_1' + Bz_2'). \end{aligned} \tag{2.4}$$

Then

$$|A(t) + B(t)| \leq |A(0) + B(0)| + 2\int_0^t \frac{p(s)}{W(s)}|h(s)|ds \tag{2.5}$$

and

$$\begin{aligned} |h(t)| &\leq \frac{1}{a(t)}|K(t, Az_1 + Bz_2, \int_0^t H(s, Az_1 + Bz_2, Az_1' + Bz_2')ds| \\ &+|Q(t, Az_1 + Bz_1, Az_1' + Bz_2')|. \end{aligned} \tag{2.6}$$

Using conditions $(b-1), (b-2)$ and $(b-3)$ we have

$$\begin{aligned} |h(t)| &\leq \frac{1}{a(t)}[(C_3(t)(|A(t)| + |B(t)|)^\alpha \int_0^t 2C_3(s)F_2(|A(s)| + |B(s)|)ds) \\ &+C_1(t)(|A(t)| + |B(t)|) + C_2(t)F_1(|A(t)| + |B(t)|)]. \end{aligned} \tag{2.7}$$

Take $|A(t)| + |B(t)| = K_1(t)$, and using (2.7), then (2.5) takes the form

$$\begin{aligned} K_1(t) &\leq K_1(0) + 2\int_0^t \frac{p(s)}{W(s)a(s)}[C_3(s)K_1^\alpha(s)(\int_0^s 2C_4(\tau)F_2(K_1(\tau))d\tau \\ &+C_1(s)K_1(s) + C_2(s)F_1(K_1(s))]ds, \end{aligned}$$

or

$$\begin{aligned} K_1(t) &\leq K_1(0) + 2\int_0^t \frac{p(s)C_3(s)K_1^\alpha(s)}{W(s)a(s)} \int_0^s 2C_4(\tau)F_2(K_1(\tau)d\tau) \\ &\quad +2\int_0^t \frac{p(s)C_1(s)}{W(s)a(s)}K_1(s)ds + 2\int_0^t \frac{p(s)C_2(s)}{W(s)a(s)}F_1(K_1(s))ds \end{aligned} \tag{2.8}$$

This is reduced by (2.1) to

$$\begin{aligned} K_1(t) &\leq K_1(0) + 2\int_0^t r(s)K_1^\alpha(s)\cdot 2\int_0^s r(\tau)F_2(K(\tau))ds \\ &+2\int_0^t r(s)K_1(s)ds + 2\int_0^t \frac{p(s)C_2(s)}{W(s)a(s)}F_1(K_1(s))ds. \end{aligned}$$

Letting $\alpha = 0$, then we have

$$\begin{aligned} K_1(t) &\leq K(0) + 2\int_0^t r(s)\int_0^s 2r(\tau)F_2(K_1(\tau))d\tau \\ &+2\int_0^t \frac{p(s)C_2(s)}{W(s)a(s)}F_1(K_1(s))ds \end{aligned}$$

Using Lemma A we have

$$\begin{aligned} K(t) &\leq \Omega^{-1}|\Omega(K_1(0) + \int_0^t \frac{2p(s)C_2(s)}{W(s)a(s)}W(1 + \int_0^s 2r(\tau)G^{-1}[C(\tau)]d\tau)ds| \\ &\times[1 + \int_0^t 2r(s)G^{-1}(C(s))ds|, \end{aligned} \tag{2.9}$$

where

$$C(t) = G(1) + \int_0^s 2r(s)ds, \Omega(r) = \int_{r_0}^r |1/W(s)|ds, r \geq r_0 > 0,$$

$$G(r) = \int_{r_0}^r \{1/s + F_2(s)\}ds, r \geq r_0 > 0, \tag{2.10}$$

and G^{-1} and Ω^{-1} as defined before.
From (2.9) and (2.10) the boundedness of $K_1(t)$ follows. Since $A(0)$ and $B(0)$ are arbitrary constants and hence can be selected as solutions of the system

$$\begin{aligned} A(0)z_1(0) + B(0)z_2(0) &= x_0, \\ A(0)z_1'(0) + B(0)z_2'(0) &= x_0', \end{aligned}$$

The limit of $A(t)$ and $B(t)$ exist as $t \to \infty$, since they are bounded. This completes the proof of the theorem.

Remark. The conclusion of the Theorem 2.1 holds if condition $(b-1)$ is replaced by

$$Q(t, x, x') \leq C_1(t)u.$$

Theorem 2.2. *If in addition to the assumptions of theorem 2.1 we further assume that z_1 and z_2 are in $L_p(0,\infty)$, (are bounded), $1 \le p \le \infty$, and*

$$C_1(t)/a(t)W(t), C_2(t)/a(t)W(t), \text{ and } [C_3(t)/a(t)W(t)]\int_0^t C_4(s)ds \tag{2.11}$$

are in $L_q(0,\infty)$, where q is the conjugate exponent of p, then the solution $x(t)$ defined by (2.2) is also bounded.

Proof. From (2.2) and (2.3) we have

$$\begin{aligned} x(t) &= A(0)z_1(t) + B(0)z_2(t) + z_1(t)\int_0^t \frac{z_2(s)}{W(s)a(s)}[(ks, A(s)z_1(s) \\ &+B(s)z_2(x), +B(s)z_2(s), \int_0^s H(\tau, A(\tau)z_1(\tau) \\ &+B(\tau)z_2(\tau), A(\tau)z_1'(\tau) + B(\tau)z_2'(\tau))d\tau]ds \\ &+z_1(t)\int_0^t \frac{z_2(s)}{W(s)a(s)}Q(s, A(s)z_1(s) + B(z)z_2(s), \\ &A(s)z_1'(s) + B(\tau)z_2'(s))ds \\ &-z_2(t)\int_0^t \frac{z_1(s)}{W(s)a(s)}[K(s, Az_1 + Bz_2, \int_0^s H(\tau, Az_1 + Bz_2, \\ &Az_1' + Bz_2')d\tau]ds \\ &-z_2(t)\int_0^t \frac{z_1(s)}{W(s)a(s)}Q(s, Az_1 + Bz_2, Az_1' + Bz_2')ds. \end{aligned}$$

Hence by using conditions $(b-a), (b-2), (b-3)$ we obtain

$$\begin{aligned} |x(t)| &\le |z_1(t)|\{|A(0)| + \int_0^t \frac{|z_2(s)|}{a(s)W(s)}C_3(s)\int_0^s |H(\tau, Az_1 + Bz_2, \\ &Az_1' + Bz_2')d\tau + C_1(s)(|A| + |B|) + C_2(s)F_1(|A| + |B|)|ds\} \\ &+|z_2(t)|\{|B(0)| + \int_0^t \frac{|z_1(s)|}{W(s)a(s)}[C_3(s)\int_0^s H d\tau \\ &+C_1(s)(|A| + |B|) + C_2(s)F_1(|A| + |B|)]ds\}. \end{aligned} \tag{2.12}$$

Since $|A(t)| + |B(t)|$ is bounded, then there exists a constant K_1 such that $|A(t)| + |B(t)| \le K_1$ for $t \ge 0$, and by using condition $(3-b)$ we obtain from (2.12).

$$|x(t)| \le |z_1(t)|\{|A(0)| + \int_0^t \frac{|z_2(s)}{a(s)W(s)}\{C_3(s)F_2(K_1)\int_0^s 2C_4(\tau)d\tau$$

$$+K_1C_1(s)+F_1(K_1)C_2(s)\}ds]+|z_2(t)|\{|B(0)|+\int_0^t \frac{|z_1(s)|}{W(s)a(s)}$$
$$\cdot\{C_3(s)F_2(K_1)\int_0^s 2C_4(\tau)d\tau+K_1C_1(s)+F_1(K_1)C_2(s)\}ds\}.$$

Let us put

$$\begin{aligned}K_2 &= |A(0)|+\|z_2\|_pF_2(K_1)\|J\|_q+K_1\|z_2\|_p\|\frac{C_1}{aW}\|_q \\ &\quad +\|z_2\|_pF_1(K_1)\|\frac{C_2}{aW}\|_q,\end{aligned}$$

and

$$\begin{aligned}K_3 &= |B(0)|+\|z_1\|_pF_2(K_1)\|J\|_q+K_1\|z_1\|_p\|\frac{C_1}{aW}\|_q \\ &\quad +\|z_1\|_pF_1(K_1)\|\frac{C_2}{aW}\|_q,\end{aligned}$$

where

$$J=\frac{C_3(t)}{a(t)W(t)}\int_0^t 2C(s)ds.$$

Consequently

$$|x(t)|\leq K_2|z_1(t)|+K_3|z(t)|. \tag{2.13}$$

Since $p\geq 1$, and the function t^p is convex, and hence

$$(\frac{\ell+m}{2})^p\leq\frac{1}{2}(\ell^p+m^p). \tag{2.14}$$

Therefore

$$|x(t)|^p\leq 2^{p-1}(K_2|z_1(t)|^p+K_3|z_2(t)|^p) \tag{2.15}$$

which implies that $\|x\|_p\leq M$, for some constant M. For the case $p=\infty$, inequality (2.13) implies that $x(t)\in L_\infty(0,\infty)$. This completes the proof.

Theorem 2.3. *Assume that conditions $(b-1)-(b-3)$ in assumption 1 hold with $F_1(u)=u^{\alpha+1}$, $F_2(u)=u$, $0\leq\alpha\leq 1$, and let condition (ii) in Theorem 2.1 be replaced by*

$$r(t), C_4(t), \text{ and } \frac{p(t)C_1(t)}{a(t)W(t)} \text{ are bounded}$$

where

$$r(t) = \max\{\frac{p(t)C_2}{a(t)W(t)}, \frac{p(t)C_3(t)}{a(t)W(t)}\}$$

and

$$|1-M^\alpha \int_0^\infty \alpha r(s)\exp(\int_0^s [\{2p(\tau)C_1(\tau)/a(\tau)W(\tau)\}+C_4(\tau))d\tau)ds]^{1/\alpha} \geq B > 0$$

for some $M > 0$ and $t \in I$. Furthermore assume that z_1 and z_2 are in $L_p(0,\infty), p \in [1,\infty]$ and condition (2.11) holds, then the conclusion of Theorem 2.2 holds.

Proof. The proof is similar to that of Theorem 2.2, and so will be omitted.

Remarks. (1) If $\alpha = 0$ and $F_2(u) = u$, then the results of Grace and Lalli [3] are included in Theorems 2.1, 2.2 and 2.3.

(2) If $K(t, x, \int_0^t H(s, x, x')ds) = 0$, and $Q(t, x, x') = C_1(t)x$, then Theorem 2.1 contains the results of Trench [7].

(3) The results obtained in this paper are generalization, to some extend, to those of Bellman [2], Mehri and Zarghame [4] and Pachpatte [6].

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