Kyungpook Mathematical Journal
Volume 32, Number 2, December, 1992

ON OSCILLATIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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The second order differential equations of the form $(py')' + qy = f$ are investigated. Sufficient conditions to ensure boundedness and oscillations of solutions are given.

I. Introduction

Consider the self-adjoint second order linear differential equation

$$[p(x)y'(x)]' + q(x)y(x) = 0 \tag{H}$$

and the corresponding nonhomogeneous equation

$$[p(x)y'(x)]' + q(x)y(x) = f(x). \tag{NH}$$

The functions p, q and f are positive continuous functions on an interval $I = [a, \infty), a \geq 0$ and their products are nondecreasing, unbounded, and of class $C'[I]$.

It is well known that if

$$\int_0^\infty [1/p(x)]dx = \infty, \quad \int_0^\infty q(x)dx = \infty \tag{1.1}$$

then every solution $y(x)$ of (H) oscillates on I.

In recent papers Maki [8], Komkov [5] showed out the usefulness of the transformation $y(x) = w(x)z(x)$ in studying qualitative properties of (H). This transformation transforms

$$y'' + q(x)y = f(x) \tag{0.1}$$

Received March 20, 1990.

into

$$(w^2z')' + w(w'' + qw)z = fw. \tag{0.2}$$

If $w(t)$ is a solution of

$$w'' + qw = 0 \tag{0.3}$$

then (0.2) becomes

$$(w^2z')' = fw. \tag{0.4}$$

Equation (0.4) enables us to study the oscillatory behaviour of solutions of (0.1) in terms of the forcing function f and the nonoscillatory solutions of (0.3).

Fink and Mary [4] proved that if the equation (H) is oscillaroty then the equation

$$(p(x)z'(x))' + \lambda q(x)z(x) = 0$$

is also oscillatory for $\lambda > 1$. It is clear that this equation can be written as

$$(p(x)z'(x)/\lambda)' + q(x)z(x) = 0.$$

Also Erbe [2], proved that multiplying the coefficient $q(x)$ by a function $a(x)$ preserves the oscillatory property. The purpose of this paper is to eatablish sufficient conditions to ensure boundedness, and oscillations of solutions of (H) and (NH). We shall make use of the following notions for every solution $y(x)$ of (H).

$$[(pq)']^+ = \{|(pq)'| + (pq)'\}/2 \qquad [(pq)']^- = \{|(pq)'| - (pq)'\}/2 \tag{1.2}$$

$$M[y, y'] = y^2 + \{(py')^2/pq\} \tag{1.3}$$

with $M_a = M[y(a), y'(a)] > 0$. Also the following theorem and definition will be used in our analysis:

Theorem A [3]. *If*

(i) $p(x) > k > 0$ *on* I *and* $\int_0^\infty (1/p(x))dx = \infty$,

(ii) $q(x) > k > 0$,

(iii) $f(x) \in L(0, \infty)$,

then all nonoscillatory solutions $y(x)$ *of (NH) satisfy* $\lim_{x\to\infty} y(x) = 0$.

Definition 1. Equation (H) is said to be disconjugate on I if no nontrivial solution of it has more than one zero.

Let $C[a,b]$ denote the set of absolutely continuous functions $r(x)$ such that $r(a)=r(b)=0$ and $|r'(x)| \in L$. Let

$$H[r;a,b] = \int_a^b (p(r')^2 - qr^2)dx \tag{1.4}$$

for all $r \in C[a,b]$. It well known, [1], that (H) is disconjugate on $|a,b|$ if and only if $J|r,a,b| > 0$ for all admissible functions $r(x)$, $(r(x) \not\equiv 0)$.

2. Main Results:

First we will establish the following theorem:

Theorem 2.1.

$$\lim_{x\to\infty} \int_a^x \{[(pg)']'/pq\}dt \tag{2.1}$$

exists and is finite, then for any solution $y(x)$ of (H)

$$\lim_{x\to\infty} \int_a^x \{(pq)'(py')^2/pq\}dt \tag{2.2}$$

exists and is finite, and all solutions $y(x)$ remain bounded.

Proof. Differentiating (1.3) and using (H) we obtain

$$(M[y,y'])' = -(pq)'(py')^2/(pq)^2. \tag{2.3}$$

Hence

$$M[y,y'] = M_a - \int_a^x (pq)'(py')^2/(pq)^2 dt. \tag{2.4}$$

Then we shall show that the expression (2.4) exists and is finite. Using (1.2), (1.3), (2.3) and (2.4) we have

$$\begin{aligned} 0 &\le (py')^2/pq \le \{(py')^2/pq\} + y^2 = M[y,y'] \\ &= M_a - \int_a^x (pq)'(py')^2/(pq)^2 dt \\ &= M_a - \int_a^x [\{(pq)'(py')^2\}^+ + \{(pq)'(py')^2\}^-]/(pq)^2 dt. \end{aligned} \tag{2.5}$$

Hence

$$(py')^2/pq \le M_a + \int_a^x \{(py')^2(pq)'\}^-/(pq)^2 dt, \tag{2.6}$$

and

$$\int_a^x \{(pq)'(py')^2\}^+/(pq)^2 dt \le M_a + \int_a^x \{(pq)'(py')^2\}^-/(pq)^2 dt. \tag{2.7}$$

From (2.6) and by using Gronwall inequality we have

$$\begin{aligned} (py')^2/pq &\le M_a \exp\left(\int_a^x \{(pq)'\}^-/pq\,dt\right) \\ &\le M_a \exp\left(\int_a^\infty \{(pq)'\}^-/pq\,dt\right) \end{aligned}$$

and thus $(py')^2/pq$ is bounded on I. It can be easily seen, from (2.7), that

$$\lim_{x\to\infty} \int_a^x \{(pq)'(py')^2\}^-/(pq)^2 dt$$
$$\lim_{x\to\infty} \int_a^x \{(pq)'(py')^2\}^+/(pq)^2 dt$$

exist and are finite and the result follows from (2.2) and (2.5).

Theorem 2.2. *If*

$$\text{(i)} \int_a^\infty (1/p)dx = \infty, \int_a^\infty q dx = \infty, \lim_{x\to\infty} p/q = 0, \tag{2.8}$$

$$\text{(ii)} \lim_{x\to\infty} \int_a^x \{(pq)'/pq\}dt, \tag{2.9}$$

exists and finite, then all solutions of (H) are bounded and oscilatory on I.

Proof. Theorem 2.1 implies that the solutions of (H) are bounded on I and $M[y, y']$ remains bounded on I. By Leighton result [6] and (1.1), it follows that under conditions (2.8) and (2.9), all solutions of (H) are oscillatory. This completes the proof.

Theorem 2.3. *A necessary and sufficient condition for solutions of (H) to be oscillatory is that there exists a function $w(x) \neq 0$ of class $C'[I]$ for which*

$$\int_a^\infty (1/pw)dx = \infty, \int_a^\infty w[(pw')' + qw]dx = \infty \tag{2.10}$$

hold.

Proof. Let $u(x)$ and $v(x)$ be any two linearly independent solution of (H) for which Abel's formula is given by

$$pW = p[uv' - u'v] = k,$$

where W is the Wronskian of u and v, and $k > 0$ is a constant. It is clear that

$$w(x) = \sqrt{u^2(x) + v^2(x)} \tag{2.11}$$

is a solution of

$$pw^3[(pw')' + qw] = k^2 \tag{2.12}$$

The transformation $y = wz$ transforms (H) into

$$(pw^2z')' + w[(pw')' + qw]z = 0 \tag{2.13}$$

Further, if $w(x)$ is such that

$$(pw^2)w[(pw')' + qw] = 1 \tag{2.14}$$

and using the transformation

$$t = \int_a^x [1/(r(t)u^2(t))]dt$$

of the independent variable, then the, two linearly independent solutions of (2.13) are

$$\sin \int_a^x [1/r(t)u^2(t)]dt, \cos \int_a^x [1/r(t)u^2(t)]dt.$$

But (2.14) is simply (12.12) with $k = 1$, thus solutions of (H) are oscillatory if and only if solutions of (2.13) are oscillatory there. It follows that if solutions of (H) are oscillatory on I, then there exists a function $w(x) \neq 0$ of class $C'[1]$ such that conditions (2.10) hold.

Conversely, if there exists function $w(x) \neq 0$ of class $C'[I]$ such the condition (2.10) hold, then by condition (1.1) and Theorem (2.2), solutions of (H) are oscillatory.

As a consequence result for equation (2.13) is following:

Theorem 2.4. *If there exists a function $w(x) > 0$ of class $C^2[I]$ such that $(pw')' + qw < 0$ for large x, then the solutions of (H) are nonoscillatory on I.*

Example 1. By taking $p(x) = 1$, $a = 1$, then equation (H) is reduced to the form

$$y'' + q(x)y = 0, 1 \leq x < \infty$$

and conditions (2.10) become

$$\int_1^\infty (1/w^2)dx = \infty, \int_1^\infty w(w^2 + qw)dx = \infty.$$

By setting $w = x^{\frac{1}{2}}$, then these conditions are reduced to

$$\int_1^\infty (1/x)dx = \infty, \int_1^\infty \{xq + |1/4x|\}dx = \infty$$

then the solutions of $y'' + q(x)y = 0$ are oscillatory on $[1, \infty)$.

Example 2. Consider the differential equations

$$y'' + (k/x^3)y = 0, 1 \leq x < \infty$$

and there exists a function z such that

$$z(x'' + (k/x^3)z) < 0, \text{ for large } x. \tag{2.15}$$

By a choice $z = x^{\frac{1}{2}}$, then the solutions of (2.15) are nonoscillatory on $[1, \infty)$.

Theorem 2.5. *If*

$$\int_1^\infty (pq)'/pq\, dx < \infty, \int_1^\infty f/(pq)^{\frac{1}{2}}\, dx < \infty \tag{2.16}$$

then all solutions of (NH) are bounded.

Proof. Using the substitution $w = y'$, then the equation (NH) takes the form

$$w' = (-p'w - qy + f)/p.$$

Define

$$E(y, w, x) = [pw^2/2q] + x^2/2.$$

Then

$$\begin{aligned} \frac{dE}{dx} &= (fw/q) - (w^2(pq)'/2q^2) \\ &\leq (fw/q) + \{(pq)'/pq\}E. \end{aligned}$$

But since

$$|fw/q| \leq (f/(pq)^{\frac{1}{2}}) \cdot \{(pw^2/2q) + \frac{1}{2}\},$$

then

$$E' = [f/(pq)^{\frac{1}{2}} + (pq)'/pq]E + f/2(pq)^{\frac{1}{2}}.$$

Using condition (2.15) and Gronwall's inequality it follows that E is bounded, and consequently $y(x)$ is bounded. This completes the proof.

Theorem 2.6. *If there exists a positive function $w(x)$ such that $w(x)f(x) \in L(I)$, $w(w'' + qw) > k_1 w^2 > k_1$ for some $k_1 > 0$ and $\int_1^\infty (1/w^2)dx = \infty$, then every nonoscillatory solution of (NH) with $p(t) = 1$ satisfies*

$$\lim_{x\to\infty} y(x)w(x) = 0.$$

Proof. The result follows from Theorem A and the hypothesis, each nonoscillatory solutions of (0.2) satisfies $\lim_{x\to\infty} z(x) = 0$.

Theorem 2.7. *If (NH) has a solution $r(x) < 0$ on I, then equation (H) is disconjugate.*

Proof. It is sufficient to show that, for any number $c > a$, equation (H) is disconjugate on $[a, c] \in I$ (i.e. $J[r, a, c] > 0$ for all $r \in C[a, b]$, $r(x) \not\equiv 0$). Let

$$w = (pr'/r)$$

then

$$\begin{aligned} w' &= [(pr')'r - p(r')^2]/r^2 = f/r - q - [(pr')^2/r^2]/p \\ &= (f/r) - q - (w^2/p). \end{aligned}$$

Hence

$$-q = w' - (w^2/p) - (f/r).$$

Therefore for any $z \in C[a, c]$ we have

$$-qz^2 = w'z^2 + (w^2z^2/p) - (fz^2/r).$$

Using (1.4) we have

$$\begin{aligned} H[z; a, c] &= \int_a^c (p(z')^2 - qz^2)dx \\ &= \int_a^c \{w'z^2 + p(z')^2 + (w^2z^2/p) - fz^2/r)\}dx \\ &= wz^2]_a^c - \int_a^c 2wzz'dx + \int_a^c \{p(z')^2 + (w^2z^2/p) - (fz^2/r)\}dx \\ &= \int_a^c \{p(z')^2 + (w^2z^2/p) - 2wzz' - (fz^2/r)\}dx \\ &= \int_a^c \{(p^{\frac{1}{2}}z' - p^{-\frac{1}{2}}wz)^2 - fz^2/r\}dx > 0. \end{aligned}$$

This completes the proof.

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