Kyungpook Mathematical Journal
Volume 32, Number 2, December, 1992

THE ASYMPTOTIC BEHAVIOR OF THE SRIVASTAVA HYPERGEOMETRIC SERIES H_C NEAR THE BOUNDARY OF ITS CONVERGENCE REGION

Megumi Saigo and H. M. Srivastava

In a series of papers by Saigo et al. ([2] to [14]), there have appeared numerous properties exhibiting the behaviors of various hypergeometric functions near the boundaries of the regions of convergence of the series defining these functions. The present paper is concerned with the triple hypergeometric series

$$(1)\quad H_C(\alpha,\beta,\beta';\delta;x,y,z) = \sum_{m,n,k=0}^{\infty} \frac{(\alpha)_{m+k}(\beta)_{m+n}(\beta')_{n+k}}{(\delta)_{m+n+k}} \frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!},$$
$$(\max[|x|,|y|,|z|]<1),$$

which was introduced by Srivastava (cf. [15]). The triple series (1) is expressed as a single series involving the Gauss series

$$(2)\ H_C = \sum_{m,n=0}^{\infty} \frac{(\beta)_{m+n}(\alpha)_m(\beta')_n}{(\delta)_{m+n}} F(\alpha+m,\beta'+n;\delta+m+n;z)\frac{x^m}{m!}\frac{y^n}{n!}.$$

We investigate the behavior of the series H_C near the side $z=1$ of the unit cube defining its convergence region. This series near the other sides $x=1$ and $y=1$ can be treated similarly. By virtue of (2) and the relation [1]

$$F(\alpha,\beta;\alpha+\beta;z) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{\{n!\}^2}(1-z)^n$$
$$(3)\qquad \cdot[2\psi(n+1)-\psi(\alpha+n)-\psi(\beta+n)-\log(1-z)],$$

Received August 17, 1991.

we have

$$(4)\quad H_C(a,b,b';a+b';x,y,1-\rho)$$
$$= \sum_{m,n=0}^{\infty} \frac{(b)_{m+n}(a)_m(b')_n}{(a+b')_{m+n}} F(a+m,b'+n;a+b'+m+n;1-\rho)\frac{x^m}{m!}\frac{y^n}{n!}$$
$$= \frac{\Gamma(a+b')}{\Gamma(a)\Gamma(b')} \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}[2\psi(k+1)$$
$$-\psi(a+m+k)-\psi(b'+n+k)-\log\rho]\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!}$$
$$= \frac{\Gamma(a+b')}{\Gamma(a)\Gamma(b')}\{2S_1-S_2-S_3-S_4\}, \text{ say}.$$

Let us first consider the series S_4. Dividing the series S_4 into two parts with $k=0$ and $k\geq 1$, we obtain

$$(5)\quad S_4 = \log\rho \sum_{m,n=0}^{\infty} (b)_{m+n}\frac{x^m}{m!}\frac{y^n}{n!}$$
$$+ab'\rho\log\rho \sum_{m,n,k=0}^{\infty} \frac{(a+1)_{m+k}(b)_{m+n}(b'+1)_{n+k}}{(a)_m(b')_n(2)_k(2)_k}\frac{x^m}{m!}\frac{y^n}{n!}\rho^k$$
$$= (1-x-y)^{-b}\log\rho + o(1), \quad (\rho\to +0).$$

To investigate the behavior of the series S_1, we use the formula

$$\psi(k+1) = -\gamma + \sum_{m=1}^{k}\frac{1}{m},$$

where γ is the Euler-Mascheroni constant, and we have

$$(6)\quad S_1 = \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!}\Big(-\gamma+\sum_{l=0}^{k-1}\frac{(1)_l}{(2)_l}\Big)$$
$$= -\gamma(1-x-y)^{-b}$$
$$+\sum_{m,n=0}^{\infty}\sum_{k=1}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}\sum_{l=0}^{k-1}\frac{(1)_l}{(2)_l}\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!} + o(1)$$
$$= -\gamma(1-x-y)^{-b} + o(1), \quad (\rho\to +0).$$

Similarly, the relation [2]

$$\psi(z+k) = \psi(z) + \sum_{m=0}^{k-1}\frac{1}{z+m}, \quad (k=1,2,3,\cdots)$$

implies that

$$
\begin{aligned}
(7)\ S_2 &= \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k} \\
&\quad \cdot(\psi(a) + \frac{1}{a}\sum_{l=0}^{m+k-1} \frac{(a)_l}{(a+1)_l})\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!} \\
&= \psi(a) \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!} \\
&\quad +\frac{1}{a} \sum_{\substack{m,n,k=0 \\ m+k\neq 0}}^{\infty} \sum_{\ell=0}^{m+k-1} \frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}\frac{(a)_l}{(a+1)_l}\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!} \\
&= \psi(a)(1-x-y)^{-b} \\
&\quad +\frac{b}{a}x\sum_{m=0}^{\infty}\sum_{l=0}^{m} \frac{(b+1)_m}{(2)_m}\frac{(a)_l}{(a+1)_l}x^m \sum_{n=0}^{\infty}\frac{(b+1+m)_n}{n!}y^n + o(1) \\
&= \psi(a)(1-x-y)^{-b} \\
&\quad +\frac{b}{a}x\sum_{p,l=0}^{\infty}\frac{(b+1)_{p+l}(a)_l}{(2)_{p+l}(a+1)_l}x^{p+1}(1-y)^{-b-1-p-l} + o(1) \\
&= \psi(a)(1-x-y)^{-b} \\
&\quad +\frac{b}{a}x(1-y)^{-b-1}F^{1:2;1}_{1:1;0}\left[\begin{matrix} b+1: & a,1; & 1; \\ 2: & a+1; & -; \end{matrix}\frac{x}{1-y},\frac{x}{1-y}\right] \\
&\quad +o(1), \quad (\rho \to +0),
\end{aligned}
$$

where we have used the binomial expansion

$$
\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}z^n = (1-z)^{-\alpha} \quad (|z|<1),
$$

and the Kampé de Fériet series $F^{1:2;1}_{1:1;0}$ is defined by

$$
(8) \qquad F^{1:2;1}_{1:1;0}\left[\begin{matrix}\alpha: & \beta,\beta'; & \eta; \\ \delta: & \epsilon; & -;\end{matrix} x,y\right] = \sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_m(\beta')_m(\eta)_n}{(\delta)_{m+n}(\epsilon)_m}\frac{x^m}{m!}\frac{y^n}{n!},
$$

which converges when $\max\{|x|,|y|\}<1$ (cf. [15] for a detailed description of the Kampé de Fériet series).

Similar arguments show that

$$
S_3 = \sum_{m,n,k=0}^{\infty}\frac{(a)_{m+k}(b)_{m+n}(b')_{n+k}}{(a)_m(b')_n(1)_k}(\psi(b') + \frac{1}{b'}\sum_{l=0}^{n+k-1}\frac{(b')_l}{(b'+1)_l})\frac{x^m}{m!}\frac{y^n}{n!}\frac{\rho^k}{k!}
$$

$$\begin{aligned}(9) \quad &= \psi(b')(1-x-y)^{-b}\\ &+\frac{b}{b'}y\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{(b+1)_n(b')_l}{(b'+1)_l(2)_n}y^n\sum_{m=0}^{\infty}\frac{(b+1+n)_m}{m!}x^m+o(1)\\ &= \psi(b')(1-x-y)^{-b}\\ &+\frac{b}{b'}y\sum_{r,l=0}^{\infty}\frac{(b+1)_{r+l}(b')_l}{(b'+1)_l(2)_{r+l}}y^{r+l}(1-x)^{-b-1-r-l}+o(1)\\ &= \psi(b')(1-x-y)^{-b}\\ &+\frac{b}{b'}y(1-x)^{-b-1}F^{1:2;1}_{1:1;0}\left[\begin{matrix}b+1: & b',1; & 1; \\ 2: & b'+1; & -;\end{matrix}\frac{y}{1-x},\frac{y}{1-x}\right]\\ &+o(1), \quad (\rho\to+0).\end{aligned}$$

Thus the relations (4) to (9) yield the following formula exhibiting the behavior of the Srivastava series near the side $z = 1$ of the unit cube defining its convergence region

$$\begin{aligned}(10) \quad & H_C(a,b,b';a+b';x,y,1-\rho)\\ &= -\frac{\Gamma(a+b')}{\Gamma(a)\Gamma(b')}\Big\{(1-x-y)^{-b}[2\gamma+\psi(a)+\psi(b')+\log\rho]\\ &+\frac{b}{a}x(1-y)^{-b-1}F^{1:2;1}_{1:1;0}\left[\begin{matrix}b+1: & a,1; & 1; \\ 2: & a+1; & -;\end{matrix}\frac{x}{1-y},\frac{x}{1-y}\right]\\ &+\frac{b}{b'}y(1-x)^{-b-1}F^{1:2;1}_{1:1;0}\left[\begin{matrix}b+1: & b',1; & 1; \\ 2: & b'+1; & -;\end{matrix}\frac{y}{1-x},\frac{y}{1-x}\right]\Big\}\\ &+o(1), \quad (\rho\to+0).\end{aligned}$$

Acknowledgement. The present investigation was initiated during the second author's visit to Fukuoka University in October 1990 while he was on subbatical leave from the University of Victoria. This work was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant OGP0007353.

References

[1] W. Magnus, F. Oberhettinger, and R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged Ed., Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[2] M. Saigo, *On a property of the Appell hypergeometric function* F_1, Math. Rep. Kyushu Univ., 12(1980), 63-67.

[3] M. Saigo, *On properties of the Appell hypergeometric functions* F_2 *and* F_3 *and the generalized Gauss function* ${}_3F_2$, Bull. Central Res. Inst. Fukuoka Univ., 66(1983), 27-32.

[4] M. Saigo, *On properties of the Lauricella hypergeometric functions* F_D, Bull. Central Res. Inst. Fukuoka Univ., 104(1988), 13-31.

[5] M. Saigo, *On properties of hypergeometric functions of three variables,* F_M *and* F_G, Rend. Circ. Mat. Palermo (2), 37(1988), 449-468.

[6] M. Saigo, *On behaviors of hypergeometric series of one, two, three and n variables near boundaries of their convergence regions of logarithmic case*, (Kyoto, 1989), Sûrikaisekikenkyûsho Kôkyûroku, 714(1990), 91-109.

[7] M. Saigo, *The asymptotic behaviors of triple hypergeometric series* F_G, F_K, F_N *and* F_R *near boundaries of their convergence regions*, (Kyoto, 1989), Sûrikaisekikenkyûsho Kôkyûroku, 714(1990), 110-126.

[8] M. Saigo, O.I. Marichev, and Nguyen Thanh Hai, *Asymptotic representations of Gaussian series* ${}_2F_1$, *Clausenian series* ${}_3F_2$ *and Appell series* F_2 *and* F_3 *near boundaries of their convergence regions*, Fukuoka Univ. Sci. Rep., 19(1989), 83-90.

[9] M. Saigo and H.M. Srivastava, *The behaviors of the Appell double hypergeometric series* F_4 *and certain Lauricella triple hypergeometric series near the boundaries of their convergence regions*, Fukuoka Univ. Sci. Rep., 19(1989), 1-10.

[10] M. Saigo and H.M. Srivastava, *The behaviors of the Lauricella hypergeometric series* $F_D^{(n)}$ *in n variables near the boundary of its convergence region*, Univ. of Victoria Rep., No.DM-480-IR, 1-23 (1988).

[11] M. Saigo and H.M. Srivastava, *The asymptotic behaviour of Lauricella's second hypergeometric series* $F_B^{(n)}$ *in n variables near the boundaries of its convergence region*, Fukuoka Univ. Sci. Rep., 19(1988), 71-82.

[12] M. Saigo and H.M. Srivastava, *The behaviors of the zero-balanced hypergeometric series* ${}_pF_{p-1}$ *near the boundary of its convergence region*, Proc. Amer. Math. Soc., 110(1990), 71-76.

[13] M. Saigo and H.M. Srivastava, *Some asymptotic formulas exhibiting the behaviors of the triple hypergeometric series* F_S *and* F_T *near the boundaries of their convergence regions*, Funk. Ekvac., 34(1991), 423-448.

[14] M. Saigo and H.M. Srivastava, *A theorem on the asymptotic behavior of Lauricella's first hypergeometric series* $F_A^{(n)}$ *near the boundary of its convergence region*, (in course of publication).

[15] H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Ltd., Chichester), John Wiley and Sons, New York-Chichester-Brisbane-Toronto, 1985.

Department of Applied Mathematics, Fukuoka University, Fukuoka 814-01, Japan.

Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada.