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DIMENSIONS OF A CANTOR TYPE SET AND ITS DISTRIBUTION SETS

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C.D. Cutler[1] and we [3] showed how to obtain the Hausdorff dimensions and packing dimensions of some subsets on **R**.

In this paper, using the strong law of large numbers with the above method, we get the Hausdorff dimensions and packing dimensions of the distribution sets which are dense in a Cantor type set. Further, we show that the Hausdorff dimensions of the distribution sets can be represented by a differentiable function of continuous parameter and that the maximum value of the dimension function is the Hausdorff dimension and packing dimension of the Cantor type set.

We review a generalized expansion of a number in $[0,1]$ ([1]). Let $0 < a_{n,1} < a_{n,2} < 1$ for any integer n. The initial proportions $a_{1,1}, a_{1,2}$ determine a division of $[0,1]$ into three disjoint intervals $[0,a_{1,1}), [a_{1,1}, a_{1,2})$ and $[a_{1,2}, 1]$. We will indicate that a point $x \in [0,1]$ falls into the ith interval ($i = 0,1,2$) by $I_1(x) = i$. Next each interval $\{x : I_1(x) = i\}$ is divided into three disjoint subintervals determined by the proportions $a_{2,1}, a_{2,2}$. This splits $[0,1]$ into 3^2 disjoint intervals ; $\{x : I_1(x) = i, I_2(x) = j\}$ $(i,j = 0,1,2)$. Each interval $\{x : I_1(x) = i, I_2(x) = j\}$ is then divided according to the proportions $a_{3,1}, a_{3,2}$. Continuing these processes, we have a generalized expansion $c(i_1, i_2, \cdots, i_n, \cdots)$ determined by n-cylinder

$$c(i_1, i_2, \cdots, i_n) = \{x : I_1(x) = i_1, I_2(x) = i_2, \cdots, I_n(x) = i_n\}.$$

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Definition 1. Let $a_{n,1} = a$ and $1 - a_{n,2} = b$ for each n. A Cantor type set C is defined as

$$C = \{x \in [0,1] : x = c(i_1, i_2, \cdots) \quad i_j \in \{0,2\}, j = 1,2,\cdots\}.$$

If $a = b$, we call C a symmetric Cantor set, otherwise a skew symmetric Cantor set [3].

Definition 2. Let $n_0(x|k)$ denote the number of times the digit 0 occurs in the first k places of the generalized expansion $c(i_1, i_2, \cdots)$ of x. For $p \in [0,1]$ we define $F(p) = \{x \in C : \lim_{k\to\infty} \frac{n_0(x|k)}{k} = p\}$. We call $F(p)$ a distribution set of C containing the digit 0 in proportion p.

Note that $F(p)$ is dense in C for each p and

$$C \supseteq \cup_{0\leq p\leq 1} F(p).$$

Let $HD(E)$ and $PD(E)$ denote the Hausdorff dimension of E and the packing dimension of E, respectively. The following proposition is well-known.

Proposition 3 ([2]). *The Cantor type set C is the invariant set with open set condition for the similarities $S_1(x) = ax$ and $S_2(x) = bx + (1-b)$. Hence $HD(C) = PD(C) = s$, where*

$$a^s + b^s = 1.$$

Proposition 4. *If we define*

$$\gamma(c(i_1, i_2, \cdots, i_k)) = \begin{cases} p^{n_0(x|k)}(1-p)^{k-n_0(x|k)} & \text{if } i_j \in \{0,2\} \text{ for } j = 1,2,\cdots,k \\ 0 & \text{otherwise} \end{cases}$$

, where $x \in c(i_1, i_2, \cdots, i_k)$, then γ extends uniquely a probability measure on the Borel sets of $[0,1]$. Further if

$$E \subset \{x \in [0,1] : \lim_{k\to\infty} \frac{\log \gamma(c_k(x))}{\log d(c_k(x))} = \theta\}$$

with $\gamma(E) > 0$, then $HD(E) = PD(E) = \theta$. (Here, $c_k(x)$ denotes the k-cylinder containing x)

Proof. It is immediate from Theorem 3.2 [1] and Corollary 4.4 [3].

Now we are ready to get the Hausdorff and packing dimension for a distribution set $F(p)$. Note that we adopt the usual convention that $0 \times \log 0 = 0$.

Theorem 5. $HD(F(p)) = PD(F(p)) = \frac{p\log p+(1-p)\log(1-p)}{p\log a+(1-p)\log b}$, *where* $0 \leq p \leq 1$.

Proof. If

$$x \in C = \{x \in [0,1] : x = c(i_1, i_2, \cdots) \quad i_j \in \{0,2\}, j = 1, 2, \cdots\},$$

then

$$\gamma(c_k(x)) = p^{n_0(x|k)}(1-p)^{k-n_0(x|k)}$$

and

$$d(c_k(x)) = a^{n_0(x|k)} b^{k-n_0(x|k)}.$$

Thus, for $x \in F(p) \subset C$,

$$\begin{aligned}\lim_{k\to\infty} \frac{\log\gamma(c_k(x))}{\log d(c_k(x))} &= \lim_{k\to\infty} \frac{n_0(x|k)\log p + [k - n_0(x|k)]\log(1-p)}{n_0(x|k)\log a + [k - n_0(x|k)]\log b} \\ &= \frac{p\log p + (1-p)\log(1-p)}{p\log a + (1-p)\log b}.\end{aligned}$$

By the strong law of large numbers, $\gamma(F(p)) = 1$.

Let $h(p)$ be the dimension function for a distribution set $F(p)$. Then $HD(F(p)) = PD(F(p)) = h(p)$ is a differentiable function of p.

We will show that there exists a proper subset $F(\frac{1}{2})$ of C which have the same dimension of C.

Theorem 6. *Let C be a symmetric Cantor set. Then*

$$\begin{aligned}h(\frac{1}{2}) &= HD(F(\frac{1}{2})) = PD(F(\frac{1}{2})) \\ &= HD(C) = PD(C).\end{aligned}$$

Proof. By Proposition 3, $HD(C) = PD(C) = \frac{-\log 2}{\log a}$. But, it follows from Theorem 5 that $HD(F(\frac{1}{2})) = PD(F(\frac{1}{2})) = \frac{-\log 2}{\log a}$ also.

Since $h(\frac{1}{2})$ is equal to $HD(C) = PD(C)$ for a symmetric Cantor set C, it is natural to ask the following question: does there exist $p_0 \in [0,1]$ such that $h(p_0) = HD(C) = PC(C)$ for a skew symmetric Cantor set C?

The following theorem gives a positive answer for the above question.

Theorem 7. *Let C be a skew symmetric Cantor set C. Then there exists $p_0 \in [0,1]$ such that*

$$h(p_0) = HD(F(p_0)) = PD(F(p_0)) = HD(C) = PD(C),$$

and $h(p_0) = \max_{0 \le p \le 1} h(p)$. Moreover, $p_0 = a^s$, where $a^s + b^s = 1$.

Proof. Plainly there exists $p_0 \in [0,1]$ such that $h(p_0) = \max_{0 \le p \le 1} h(p)$ and $h'(p_0) = 0$.

Now, suppose $h'(p) = 0$. Then $\log \frac{p}{q} \log a^p b^q = \log p^p q^q \log \frac{a}{b}$, where $q = 1 - p$. Noting $\frac{a}{b} \neq 1$ and $a^p b^p \neq 1$, we have $\frac{\log p^p q^q}{\log a^p b^q} = \frac{\log \frac{p}{q}}{\log \frac{a}{b}}$. That is, $h(p) = \frac{\log \frac{p}{q}}{\log \frac{a}{b}}$.

Put $\frac{\log \frac{p}{q}}{\log \frac{a}{b}} = s$ and $\alpha = \frac{p}{q}$. Then $(\frac{a}{b})^s = \alpha$, so $a^s = \alpha b^s$. Note that $a^s + b^s = 1 \Leftrightarrow a^s + b^s = (\alpha + 1) b^s = 1 \Leftrightarrow b^s = q$. Thus we only need to show that $b^s = q$. Since $(a^p b^q)^s = p^p q^q$, $p^p q^q = (a^s)^p (b^s)^q = (\alpha b^s)^p (b^s)^q$ $= \alpha^p b^s$. Hence $(\frac{p}{q})^p b^s = p^p q^q$, i.e., $b^s = q$. Therefore $p = a^s$.

Note 8. Let C be a Cantor type set. It seems to be more convenient for a numerical method to use $h(p)$ instead of the equation $a^s + b^s = 1$ for finding the dimension of C.

Remark 9. Let C be a Cantor type set with its distribution sets $F(p)$'s and let $\mathcal{H}^s(\mathcal{P}^s)$ denote s-dimensional Hausdorff (packing) measure. Then it would be interesting to compare the values of $\mathcal{H}^s(F(p))$ and $\mathcal{P}^s(F(p))$ for $s = HD(F(p)) = PD(F(p))$.

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