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L^2-KILLING p-FORMS ON A COMPLETE, NON-COMPACT RIEMANNIAN MANIFOLD WITHOUT BOUNDARY

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1. Introduction

The results of the study of killing tensor fields on a compact Riemannian manifold without boundary had been listed in Yano's book [19]. In [23], non-existence of L^2-Killing vector fields on a complete Riemannian manifold without boundary was discussed.

The study of L^2-harmonic p-forms on a complete Riemannian manifold has been done in [6], [7] and [21].

The purpose of the present paper is to investigate the properties and non-existence of L^2-killing p-forms on a complete, non-compact Riemannian manifold without boundary.

We shall be in C^∞-category. Latin indicies run from 1 to n. The Einstein summation convention will be used.

2. Preliminaries

Let M be an orientable Riemannian manifold of dimension n and g (resp. ∇) the Riemannian metric (resp. the Riemannian connection) on M.

We consider a p-form on M

$$\omega = \frac{1}{p!}\omega_{i_1,\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{2.1}$$

or a skew-symmetric tensor field of type $(0,p)$. Then $w_{i_1\cdots i_p}$ are local components of the p-form ω. The *exterior differential* $d\omega$ of a p-form ω on

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M is a $(p+1)$-form given by

$$\begin{aligned}(2.2)\qquad d\omega \;=\;& \frac{1}{(p+1)!}\Big\{\nabla_i\omega_{i_1,\cdots i_p} - \nabla_{i_1}\omega_{ii_2\cdots i_p} - \cdots \\ & -\nabla_{i_p}\omega_{i_1\cdots i_{p-1}i}\Big\}dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.\end{aligned}$$

From a p-form ω on M, the $(p-1)$-form given by

$$(2.3)\qquad \delta\omega = -\frac{1}{(p-1)!}(g^{ji}\nabla_j\omega_{ii_2\cdots i_p})dx^{i_2} \wedge \cdots \wedge dx^{i_p}$$

is called the *codifferential* of the p-form ω. If ω is a function on M, then we put $\delta\omega = 0$. For any p-form ω on M, it is clear that

$$(2.4)\qquad d(d\omega) = 0 \quad \text{and} \quad \delta(\delta\omega) = 0.$$

The *Laplace-Beltrami operator* $\Delta = \delta d + d\delta$ is represented by

$$\begin{aligned}\Delta\omega \;=\;& \delta d\omega + d\delta w \\ (2.5)\qquad \;=\;& -\frac{1}{p!}\{g^{ji}\nabla_j\nabla_i\omega_{i_1\cdots i_p} - \sum_{s=1}^{p} K_{i_s}{}^t\omega_{i_1\cdots t\cdots i_p} \\ & -\sum_{t<s}^{1\cdots p} K_{i_t i_s}{}^{ab}\omega_{i_1\cdots a\cdots b\cdots i_p}\}dx^{i_1} \wedge \cdots \wedge dx^{i_p}\end{aligned}$$

by the Ricci identity and $g^{ji}K_{jsi}{}^t = -K_s{}^t$ for any p-form ω, where $K_{kji}{}^h$ and K_{ji} are local components of the Riemannian curvature tensor and Ricci tensor of M, respectively.

Let r be the scalar curvature of the Riemmanian manifold M, that is, $r = g^{ji}K_{ji}$. If the Riemannian manifold is of constant curvature, then we have

$$(2.6)\qquad K_{kjih} = \frac{1}{n(n-1)}(g_{kh}g_{ji} - g_{jh}g_{ki}), \quad K_{ji} = \frac{r}{n}g_{ji}.$$

Let M be an n-dimensional Riemannian manifold with the fundamental metric g. The change of the metric $g^* = \rho^2 g$, where ρ is a certain positive function, does not change the angle between two vectors at a point and so is called a *conformal transformation* of the metric.

Let C be the Weyl conformal curvature tensor of M with components $C_{kji}{}^h$. Then we have

$$(2.7)\qquad C_{kji}{}^h = K_{kji}{}^h + \delta_k^h C_{ji} - \delta_j^h C_{ki} + C_k{}^h g_{ji} - C_j{}^h g_{ki},$$

where

$$C_{ji} = -\frac{1}{n-2}K_{ji} + \frac{1}{2(n-1)(n-2)}rg_{ji}$$

and

$$C_k{}^h = C_{ki}g^{ih}.$$

If a Riemannian metric g is conformally related to a Riemannian metric g^* which is locally flat, then the Riemannian manifold M with the metric g is said to be *conformally flat.* If a Riemannian manifold M is conformally flat, then $C = 0$ for $\dim M > 3$ (cf. [4], [15], [16]). Hence, for a conformally flat Riemannian manifold M of $\dim M > 3$, we have

$$\begin{aligned}(2.8) \qquad K_{kjih} &= \frac{1}{n-2}(g_{kh}K_{ji} - g_{jh}K_{ki} + g_{ji}K_{kh} - g_{ki}K_{jh}) \\ &\quad -\frac{r}{(n-1)(n-2)}(g_{kh}g_{ji} - g_{jh}g_{ki}).\end{aligned}$$

From now on we assume that M is a complete, non-compact, connected and orientable Riemannian manifold of dimension n without boundary unless special mention.

3. L^2-p-forms on M

In this section, we will introduce special Lipschitz continuous functions and study some properties of L^2-p-forms on M.

Let $\Lambda^P(M)$ be the space of all p-forms on M and $\Lambda^P_o(M)$ the subspace of $\Lambda^P(M)$ composed of forms with compact supports.

The Hodge $*$-operator on $\Lambda^P(M)$ is defined by (cf. [7],[8], [12],[13])

$$\begin{aligned}(3.1) \quad *\omega &= \sum_{j_1<\cdots<j_p;k_1<\cdots<k_{n-p}} g^{i_1j_1}\cdots g^{i_pj_p} \\ &\quad \times\delta^{1\cdots\cdots\cdots\cdots n}_{j_1\cdots j_pk_1\cdots k_{n-p}}\sqrt{det(g_{ji})} \times \omega_{i_1\cdots i_p}dx^{k_1}\wedge\cdots\wedge dx^{k_{n-p}},\end{aligned}$$

where $\omega = \frac{1}{p!}\omega_{i_1\cdots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$ and $\delta^{1\cdots\cdots\cdots\cdots n}_{j_1\cdots j_pk_1\cdots k_{n-p}}$ denotes the Kronecker symbol. Thus we may define a global scalar product $\ll\ ,\ \gg$ on $\Lambda^P_0(M)$ by (cf. [5], [11], [12], [22])

$$(3.2) \qquad \ll\phi,\psi\gg = \int_M <\phi,\psi> dV = \int_M \phi\wedge *\psi.$$

Also, we have (cf. [6],[8],[24])

$$(3.3) \qquad \ll d\phi,\psi\gg = \ll\phi,\delta\psi\gg$$

for any $\phi \in \Lambda_0^P(M)$ and $\psi \in \Lambda_0^{P+1}(M)$.

Let x_0 be a fixed point of M and $\rho(p)$ the distance from x_0 to $p \in M$. Then the set

$$B(2\alpha) = \{p \in M| \quad \rho(p) \leq 2\alpha\} \tag{3.4}$$

is compact in M for any $\alpha > 0$.

On the other hand, if we consider a cut-off function μ on R satisfying (cf. [11],[20])

$$\begin{cases} 0 \leq \mu \leq 1 & \text{on } R \\ \mu(y) = 1 & \text{for } y \leq 1 \\ \mu(y) = 0 & \text{for } y \geq 2, \end{cases} \tag{3.5}$$

then we can define a family $\{\lambda_\alpha\}$ of Lipschitz continuous functions on M by (cf. [9], [20], [22])

$$\lambda_\alpha(p) = \mu(\rho(p)/\alpha), \quad \alpha = 1, 2, 3 \cdots \tag{3.6}$$

for any $p \in M$. Thus the family $\{\lambda_\alpha\}$ satisfies the following properties:

$$\begin{cases} 0 \leq \lambda_\alpha(p) \leq 1 & \text{for any } p \in M \\ \operatorname{supp}\lambda_\alpha \subset B(2\alpha) & \\ \lambda_\alpha(p) = 1 & \text{for any } p \in B(\alpha) \\ \lim_{\alpha \to \infty} \lambda_\alpha = 1 & \\ |d\lambda_\alpha| \leq D\alpha^{-1} & \text{almost everywhere on } M, \end{cases} \tag{3.7}$$

where D is a positive constant independent on α (cf. [1],[3], [7],[8],[22],[23],[25]).

In fact, ρ is locally Lipschitz function and $|d\rho|^2 \leq n$. Since $d\lambda_\alpha = \frac{1}{\alpha}(d\mu/dt)d\rho$ at the point where the derivative of ρ exists (cf. [10]), setting $A := \sup|\frac{d\mu}{dt}|$ implies (cf. [2],[8]).

Lemma 3.1. *Under the above notations,*

$$\|d\lambda_\alpha \wedge \omega\|^2_{B(2\alpha)} \leq \frac{nA^2}{\alpha^2}\|\omega\|^2_{B(2\alpha)}, \tag{3.8}$$

for any $\omega \in \Lambda_0^P(M)$, where A is a positive constant depending only on μ and $\|\omega\|^2_{B(2\alpha)} = \ll w, w \gg_{B(2\alpha)} = \int_{B(2\alpha)} < \omega, \omega > *1$.

Let $L_P^2(M)$ be the completion of $\Lambda_0^P(M)$ with respect to the global scalar product $\ll \, , \gg$. A tensor field $\omega \in L_P^2(M) \cap \Lambda^P(M)$ is called the L^2-p-form on M. Then we remark that $\|w\| < \infty$, $\lambda_\alpha\omega \in \Lambda_0^P(M)$ and $\lambda_\alpha\omega \to \omega$ as $\alpha \to \infty$ in the strong sense for any L^2-p-form ω on M (cf. [2], [9], [22]).

On the other hand, for any L^2-p-form ω on M, it is clear that (cf. [2], [9])

$$(3.9)\ | \ll d\omega, 2\lambda_\alpha d\lambda_\alpha \wedge \omega \gg_{B(2\alpha)} | \leq \frac{nA^2}{\alpha^2}(\|\lambda_\alpha d\omega\|^2_{B(2\alpha)} + \|\omega\|^2_{B(2\alpha)}).$$

4. Non-existence of L^2-Killing p-forms on M

In this section, we will find useful properties of L^2-Killing p-form on M. From these properties, we will obtain our main result which is a natural extension of that of K. Yano ([18], [19]) in the case of compact Riemannian manifold. Furthermore, we will study relations between curvature (or Ricci) tensor and non-existence of L^2-Killing p-forms on M.

A p-form ω on M is a *Killing p-form* if it satisfies

$$(4.1) \qquad (p+1)\nabla\omega = d\omega.$$

For a Killing p-form ω, it is clear that $\nabla\omega$ is a skew-symmetric and $\delta\omega = 0$.

We first introduce the following lemma due to T. Takahashi ([14]).

Lemma 4.1. *A p-form ω on M is a Killing form if and only if*

$$(4.2) \qquad \triangle\omega = (p+1)\delta d\omega \quad \text{and} \quad \delta\omega = 0$$

or

$$(4.3)\ g^{ji}\nabla_j\nabla_i\omega_{i_1\cdots i_p} + \frac{1}{p}\sum_{s=1}^{p} K_{i_s}{}^{t}\omega_{i_1\cdots t\cdots i_p} + \frac{1}{p}\sum_{t<s}^{1\cdots p} K_{i_t i_s}{}^{ab}\omega_{i_1\cdots a\cdots b\cdots i_p} = 0$$

and

$$(4.4) \qquad g^{ji}\nabla_j\omega_{ii_2\cdots i_p} = 0.$$

We consider (the square length) $|\lambda_\alpha\omega|^2$ of a L^2-p-form ω on M. Then we have

$$(4.5) \qquad \frac{1}{2}\triangle|\lambda_\alpha\omega|^2 = < \delta\nabla(\lambda_\alpha\omega), \lambda_\alpha\omega > - |\nabla(\lambda_\alpha\omega)|^2.$$

Suppose that ω is a L^2-Killing p-form, then we have

$$(4.6) \qquad (\delta\nabla\omega)_{i_1\cdots i_p} = \frac{1}{p}\sum_{s=1}^{p} K^t_{i_s}\omega_{i_1\cdots t\cdots i_p} + \frac{1}{p}\sum_{t<s}^{1\cdots p} K_{i_t i_s}{}^{ab}\omega_{i_1\cdots a\cdots b\cdots i_p},$$

and consequently

$$(4.7) \qquad < \delta\nabla\omega, \omega >= F_p(\omega, \omega),$$

where the quadratic form $F_p(\omega,\omega)$ is given by

$$(4.8) \qquad \begin{aligned} F_p(\omega,\omega) &= K_{ji}\omega^j{}_{i_2\cdots i_p}\omega^{ii_2\cdots i_p} \\ &\quad + \frac{p-1}{2}K_{kjih}\omega^{kj}{}_{i_3\cdots i_p}\omega^{ihi_3\cdots i_p}. \end{aligned}$$

Since $F_p(\omega,\omega)$ is bilinear, $< \lambda_\alpha\delta\nabla\omega, \lambda_\alpha\omega >= F_p(\lambda_\alpha\omega, \lambda_\alpha\omega)$.

Thus we have

Lemma 4.2. *If ω is a L^2-Killing p-form on M, then it holds that*

$$(4.9) \quad \begin{aligned} \frac{1}{2}\Delta|\lambda_\alpha\omega|^2 &= F_p(\lambda_\alpha\omega, \lambda_\alpha\omega) - \lambda_\alpha^2|\nabla\omega|^2 - |d\lambda_\alpha \wedge \omega|^2 \\ &\quad -2 < \lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega > + < \delta d\lambda_\alpha, \lambda_\alpha|\omega|^2 > . \end{aligned}$$

Using Stokes' theorem and Lemma 4.2, we have our main results:

Theorem 4.3. *Let M be a complete, non-compact, connected and orientable Riemannian manifold of dimension n without boundary. If the quadratic form $F_p(\omega,\omega)$ is negative-semidefinite, then any L^2-Killing p-form on M is parallel.*

Proof. By Stokes' theorem, we have

$$(4.10) \qquad \frac{1}{2}\int_{B(2\alpha)}\Delta|\lambda_\alpha\omega|^2 dV = -\frac{1}{2}\int_{\partial B(2\alpha)} < N, d|\lambda_\alpha\omega|^2 > dB,$$

where N is the outer normal vector to $\partial B(2\alpha)$ and dB is the volume element of $\partial B(2\alpha)$.

Since $\partial B(2\alpha) = \partial M \cup \{p \in M | \rho(p) = 2\alpha\}$, $\lambda_\alpha = 1$ on ∂M and $\lambda_\alpha = 0$ on $\{p \in M | \rho(p) = 2\alpha\}$ (cf. [2]). Moreover, since $\partial M = \phi$, the right hand side of (4.10) is equal to zero. Thus from Lemma 4.2, we have

$$\begin{aligned} 0 &\leq |\int_{B(2\alpha)} F_p(\lambda_\alpha\omega, \lambda_\alpha\omega)dV - \int_{B(2\alpha)} \lambda_\alpha^2|\nabla\omega|^2 dV| \\ &= |\int_{B(2\alpha)} |d\lambda_\alpha \wedge \omega|^2 + 2\int_{B(2\alpha)} < \lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega > dV \\ &\quad - \int_{B(2\alpha)} < \delta d\lambda_\alpha, \lambda_\alpha|\omega|^2 > dV| \\ &\leq \|d\lambda_\alpha \wedge \omega\|^2_{B(2\alpha)} + 2 \ll \lambda_\alpha\nabla\omega, 2d\lambda_\alpha \wedge \omega \gg_{B(2\alpha)} \\ &\quad + \ll d\lambda_\alpha, d(\lambda_\alpha|\omega|^2) \gg_{B(2\alpha)} . \end{aligned}$$

Letting $\alpha \to \infty$, from Lemma 3.1 we have

$$\int_M F_p(\omega,\omega)dV = \int_M |\nabla\omega|^2 dV.$$

Hence if $F_p(\omega,\omega)$ is negative-semidefinite, then $\nabla\omega = 0$.

Corollary 4.4. *If the quadratic form $F_p(\omega,\omega)$ is negative-semidefinite on M and $F_p(\omega,\omega) < 0$ for some point in M, then there are no non-zero L^2-Killing p-forms on M.*

Now, we can find an example for our main results.

Example 4.5. We set $r = (x^2+y^2+z^2)^{\frac{1}{2}}$ for any point (x,y,z) of R^3 and

$$x = r\cos\theta_1 \quad y = r\sin\theta_1\cos\theta_2 \quad z = r\sin\theta_1\sin\theta_2,$$

that is, (θ_1,θ_2,r) is the spherical coordinates in R^3. For two positive constant numbers a_1 and a_2 $(a_1 < a_2)$, we consider a metric ds^2 on R^3 such that

$$\begin{aligned} ds^2 &= r^{-\frac{2}{3}}\{(d\theta_1)^2 + \sin^2\theta_1(d\theta_2)^2\} + (dr)^2 \quad \text{for } r \geq \frac{a_1+a_2}{2}, \\ ds^2 &= r^2\{(d\theta_1)^2 + \sin^2\theta_1(d\theta_2)^2\} + (dr)^2 \quad \text{for } r \leq a_1. \end{aligned}$$

Then $\mathcal{M} = (R^3, ds^2)$ is a complete, non-compact, connected and orientable Riemannian manifold. We set $M = \{(\theta_1,\theta_2,r) \in \mathcal{M} | r > a_2\}$, then M is a non-compact, connected and orientable Riemannian manifold without boundary.

Thus we have

(1) The volume of M is infinite.

(2) $\omega = \frac{2}{r}\sin\theta_1 d\theta_1 \wedge d\theta_2$ is a L^2-Killing 2-form on M.

(3) $F_2(\omega,\omega) = -r^{-\frac{8}{3}} \to 0$ as $r \to \infty$.

Remark. For the case of $p = 1$, we can take an example for our main results as in S. Yorozu ([23]).

From now on, we would like to investigate relations between curvature (or Ricci) tensors and non-existence of L^2-Killing p-forms on M.

Firstly, we consider that the Riemannian manifold M of dimension n is of constant curvature. Then from (2.6), we have

$$F_p(\omega,\omega) = \frac{n-p}{n(n-1)} r|\omega|^2, \tag{4.11}$$

from which together with Corollary 4.4, we have

Proposition 4.6. *Let M be as Theorem 4.3. If M is of negative constant curvature, then there are no non-zero L^2-Killing p-forms on $M(p = 1, \cdots, n-1)$.*

Secondly, we assume that M is conformally flat Riemannian manifold of dimension $n > 3$. Then from (2.8), we have

$$F_p(\omega,\omega) = \frac{n-2p}{n-2} K_{ji}\omega^j{}_{i_2\cdots i_p}\omega^{ii_2\cdots i_p} + \frac{p-1}{(n-1)(n-2)} r|\omega|^2. \tag{4.12}$$

On the other hand, we assume that $R(v,v)$ is negative-definite and denote by $-L$ the largest (negative) eigenvalue of the matrix (K_{ji}). Then we have

$$R(v,v) \leq -Lg(v,v), \quad r \leq -nL < 0, \tag{4.13}$$

which and (4.12) imply

$$F_p(\omega,\omega) \leq -\frac{n-p}{n-1} L|\omega|^2. \tag{4.14}$$

Thus, form Corollary 4.4 we have

Proposition 4.7. *Suppose that M is a complete, non-compact and connected conformally flat orientable Riemannian manifold of dimension $n > 3$ without boundary. If the Ricci curvature of M is negative-semidefinite and the Ricci curvature is negative for some point, then there are no non-zero L^2-Killing p-forms on $M(p = 1, \cdots, [\frac{n}{2}])$.*

Finally, we suppose that the curvature tensor K_{kjih} of M satisfies the inequalities

$$-b \leq -\frac{K_{kjih}\omega^{kj}\omega^{ih}}{\omega_{ji}\omega^{ji}} \leq -\frac{1}{2}b < 0 \tag{4.15}$$

for a certain negative number $-b$ and for any skew-symmetric tensor field ω of type $(0,2)$.

Taking two mutually orthogonal unit vectors X and Y and putting

$$\omega^{ji} = Y^jX^i - Y^iX^j,$$

we obtain

$$-\frac{1}{2}b \leq -K_{kjih}Y^kX^jY^iX^h \leq -\frac{1}{4}b < 0. \tag{4.16}$$

Put $X_1 = X$ and take $n-1$ unit vecotrs $X_2, \cdots, X_n$ which are orthogonal to X_1 and to each other. Then we have

$$\sum_{s=1}^{n} X_s^j X_s^i = g^{ji}.$$

On the other hand, from (4.16) we have

$$(4.17) \qquad -\frac{1}{2}b \leq -K_{kjih} X_s^k X^j X_s^i X^h \leq -\frac{1}{4}b < 0.$$

and consequently

$$(4.18) \qquad \begin{cases} -\frac{n-1}{2}b \leq R(X,X) \leq -\frac{n-1}{4}b < 0, \\ F_p(\omega,\omega) \leq -\frac{n-2p+1}{4}b|\omega|^2. \end{cases}$$

Hence, from Corollary 4.4 we have

Proposition 4.8. *Let M be as Theorem 4.3. If the curvature tensor satisfies (4.15), then there are no non-zero L^2-Killing p-forms on $M(p = 1, \cdots, [\frac{n}{2}])$.*

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