Kyungpook Mathematical Journal
Volume 32, Number 1, June, 1992

REGULAR EXTENDED TRIPLE SYSTEMS

Chung Je Cho

1. Introduction

An *extended triple system* of order v is a pair (V, B) where V is a v-set and B is a collection of nonordered triples of elements in V (called blocks), where each triple may have repeated elements, such that every pair of elements of V, not necessarily distinct, belongs to exactly one block. The blocks of B are of three types:

$$\{a, a, a\}, \quad \{b, b, c\}, \quad \{x, y, z\}$$

where the element a is called an idempotent and b a nonidempotent of the system. We will denote by $E(v; n)$ an extended triple system of order v, which has n idempotents. It is straighforward to see that if there exists an $E(v; n)$, then $n = 0, 1, \cdots$, or v. Johnson and Mendelsohn [3] obtained a necessary condition for the existence of an $E(v; n)$, and Bennett and Mendelsohn [1] showed that this necessary condition was also sufficient.

Theorem 1.1 [1,3]. *There exists an $E(v; n)$ if and only if*
(i) $v \equiv 0 (\text{mod} 3)$ *and* $n \equiv 0 (\text{mod} 3)$ *or*
(ii) $v \equiv 1$ *or* $2 (\text{mod} 3)$ *and* $n \equiv 1 (\text{mod} 3)$ *or*
(iii) v *is even and* $n \leq v/2$ *or*
(iv) $n = v - 1$ *and* $v = 2$.

An *automorphism* of an $E(v; n)$ (V, B) is a permutation of V which fixes B setwise, i.e. $\alpha(B) = B$ where $\alpha(X) = \{\alpha(x) | x \in X\}$.

Let α be a permutation of degree v and type $[\alpha] = [\alpha_1, \alpha_2, \cdots, \alpha_v]$, i.e. the disjoint cycle decomposition of α consists of α_i cycles of length i and

Received July 10, 1991.
This research is supported by a grant from the Korea Science and Engineering Foundation Grant 1990-1991.

$\sum i\alpha_i = v$. An $E(v;n)$ admitting α as an automorphism will be denoted by $E_\alpha(v;n)$. If $[\alpha] = [0,0,\cdots,0,1]$, then $E_\alpha(v;n)$ is called *cyclic.* If $[\alpha] = [1,0,\cdots,k,0,\cdots,0]$, i.e. $\alpha_1 = 1$, $\alpha_{(v-1)/k} = k$ and $\alpha_i = 0$ otherwise, then an $E_\alpha(v;n)$ is called *k-rotational.* If $[\alpha] = [0,0,\cdots,0,k,0,\cdots,0]$, i.e. $\alpha_{v/k} = k$ and $\alpha_i = 0$ otherwise, then an $E_\alpha(v;n)$ is called *k-regular.*

Theorem 1.2 [see 2]. *There exists a cyclic $E(v;n)$ if and only if*
(i) $n = v$ *and* $v \equiv 1$ *or 3* $(\mathrm{mod}6), v \neq 9$ *or*
(ii) $n = 0$ *and* $v \equiv 3(\mathrm{mod}6)$.

Theorem 1.3 [2]. *(1) There exists a 1-rotational $E(v;n)$ if and only if*
(i) $n = v$ *and* $v \equiv 3$ *or* $9(\mathrm{mod}24)$ *or*
(ii) $n = 1$ *and* $v \equiv 1$ *or* $2(\mathrm{mod}3), v \neq 10$.
(2) *There exists a 2-rotational $E(v;n)$ if and only if*
(i) $n = v$ *and* $v \equiv 1,3,7,9,15$ *or* $19(\mathrm{mod}24)$ *or*
(ii) $n = (v+1)/2$ *and* $v \equiv 1(\mathrm{mod}6)$ *or*
(iii) $n = 1$ *and* $v \equiv 1$ *or 5* $(\mathrm{mod}6)$.
(3) *A necessary and sufficient condition for the existence of a 3-rotational $E(v;n)$ is*
(i) $n = v$ *and* $v \equiv 1$ *or 19* $(\mathrm{mod}24)$ *or*
(ii) $n = 1$ *and* $v \equiv 1(\mathrm{mod}3)$ *or*
(iii) $n = (v+2)/3$ *or* $(2v+1)/3$ *and* $v \equiv 1(\mathrm{mod}18)$, *except possibly for* $v \equiv 37$ *or 55* $(\mathrm{mod}72)$ *and* $n = (v+2)/3$ *or* $(2v+1)/3$.

A *Steiner triple system* of order v, denoted by $STS(v)$, is a pair (V,B) where V is a v-set and B is a set of 3-subsets of V, called blocks, such that each 2-subset of V belongs to precisely one block. It is well-known that a $STS(v)$ exists if and only if $v \equiv 1$ or $3(\mathrm{mod}6)$, Peltesohn [5] has shown that a cyclic $STS(v)$ exists if and only if $v \equiv 1$ or $3(\mathrm{mod}6)$, $v \neq 9$; and then it is easily seen that a 3-regular $STS(v)$ exists if and only if $v \equiv 3(\mathrm{mod}6)$.

An (A,k)-*system* (a (B,k)-*system*) is a set of ordered pairs $\{(a_r,b_r) \mid r = 1,2,\cdots,k\}$ such that $b_r - a_r = r$ for $r = 1,2,\cdots,k$, and $\cup_{r=1}^{k}\{a_r,b_r\} = \{1,2,\cdots,2k\}$ $(= \{1,2,\cdots,2k-1,2k+1\})$. An (A,k)-system and a (B,k)-system are essentially the same as a Skolem k-sequence and a hooked Skolem k-sequence, respectively [4,6]. It is well-known that an (A,k)-system and a (B,k)-system exist if and only if $k \equiv 0$ or 1 $(\mathrm{mod}4)$ and $k \equiv 2$ or 3 $(\mathrm{mod}4)$, respectively [see 4,6,7].

In this paper, we obtain a necessary and sufficient condition for the existence of k-regular $E(v;n)'s$ with $k \leq 3$, and a necessary condition for the existence of 4-regular $E(v;n)'s$ and also show that this necessary condition is sufficient, except possibly for $v \equiv 12$ or $20 \pmod{24}$ and $n = v/2$.

2. Regular Extended Triple Systems

If (V, B) is a regular $E(v;n)$ with automorphism α, then B can be partitioned into disjoint orbits under α, i.e. an orbit of a block $\{a, b, c\}$ under α is the set of blocks $\{\alpha^i(a), \alpha^i(b), \alpha^i(c)\}$ where $0 \leq i \leq v$. Thus a collection of blocks taken from each orbit precisely once, called base blocks, represents the whole blocks B.

Throughout, we will assume that our k-regular $E(v;n)'s$ are based on $V = Z_{v/k} \times Z_k$, where Z_m is the additive group of all integers modulo v with residue classes $\{0, 1, \cdots m-1\}$, and the corresponding k-regular automorphism is $\alpha = (0_0 1_0 \cdots (v/k-1)_0)\ (0_1 1_1 \cdots (v/k-1)_1) \cdots (0_{k-1} 1_{k-1} \cdots (v/k-1)_{k-1})$; here instead of (x, i) we write for brevity x_i unless other specified. By simple arguments, we have easily seen the following lemma.

Lemma 2.1. *If there exists a k-regular $E(v;n)$ with $k > 1$, then*
(i) *v/k must be an odd integer, and*
(ii) *$n = t(v/k)$ for $t = 0, 1, \cdots,$ or k.*

Since a cyclic $E(v;n)$ is also considered as a 1-regular system, we start with $k = 2$. By Lemma 2.1, if there exists a 2-regular $E(v;n)$ then $n = 0, v/2$ or v. But since $v/2$ is an integer, v must be even; so $n \leq v/2$ by Theorem 1.1. Thus $n = 0$ or $v/2$.

Lemma 2.2. *If there exists a 2-regular $E(v;n)$, then*
(i) *$n = 0$ and $v \equiv 6 \pmod{12}$ or*
(ii) *$n = v/2$ and $v \equiv 2$ or $6 \pmod{12}$.*

Proof. (i) If $n = 0$, then $v \equiv 0 \pmod 3$ by Theorem 1.1. But we know that $v/2$ is an odd integer and hence $v \equiv 6 \pmod{12}$.

(ii) If $n = v/2$ then $v/2 \equiv 0$ or $1 \pmod 3$ by Theorem 1.1, and hence $v \equiv 0$ or $2 \pmod 6$; so $v \equiv 2$ or $6 \pmod{12}$ since $v/2$ is odd.

Lemma 2.3. *There exists a 2-regular $E(18;0)$.*

Proof. Base blocks: $B = B_1 \cup B_2$ where

$$B_1 \ : \ \{\{0_0, 0_0, 0_1\}, \{0_1, 0_1, 2_1\}, \{0_1, 1_1, 4_1\}\},$$

$$B_2 \quad : \quad \{\{0_0, r_0, (b_r)_1\} \mid r = 1,2,3,4\}$$

where $\{(a_r, b_r)|r = 1,2,3,4\}$ is an $(A,4)$-system. Then (V,B) is a 2-regular $E(18;0)$.

Lemma 2.4. *If* $v \equiv 6 (\mathrm{mod}\, 12)$, *then there exists a 2-regular* $E(v;0)$.

Proof. The case $v = 18$ has been treated in Lemma 2.3. Let $v = 12t+1$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$$B_1 \quad : \quad \begin{cases} \{\{0_0, 0_0, 0_1\}, \{0_1, 0_1, (2t+1)_1\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{0_0, 0_0, (6t+2)_1\}, \{0_1, 0_1, (2t+1)_1\}\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

$$B_2 \quad : \quad \{\{0_0, r_0, (b_r)_1\} | r = 1, 2, \cdots, 3t+1\}$$

where $\{(a_r, b_r)|r = 1, 2, \cdots 3t+1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$.
B_3: a set of base blocks which form a cyclic $STS(6t+3)$, except the base block $\{0_1, (2t+1)_1, (4t+2)_1\}$, based on $Z_{6t+3} \times \{1\}$. Then (V,B) is a 2-regular $E(v;0)$.

Thus, we have the following theorem.

Theorem 2.5. *There exists a 2-regular* $E(v;n)$ *if and only if* $v \equiv 6$ *(mod 12).*

Lemma 2.6. *If* $v \equiv 2$ *(mod 12), then there exists a 2-regular* $E(v; v/2)$.

Proof. Let $v = 12t+2$. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$$B_1 \quad : \quad \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, 0_0\}\} & \text{if } t \equiv 0 \text{ or } 3 \pmod 4; \\ \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (6t)_0\}\} & \text{if } t \equiv 1 \text{ or } 2 \pmod 4, \end{cases}$$

B_2 : a set of base blocks which form a cyclic $STS(6t+1)$ based on $Z_{6t+1} \times \{0\}$,

$$B_3 \quad : \quad \{\{0_1, r_1, (b_r)_0\} | r = 1, 2, \cdots, 3t\}$$

where $\{(a_r, b_r)|r = 1, 2, \cdots, 3t\}$ is an $(A, 3t)$-system or a $(B, 3t)$-system depending on whether $t \equiv 0, 3 \pmod 4$ or $t \equiv 1, 2 \pmod 4$. Then (V,B) is a 2-regular $E(v; v/2)$.

Lemma 2.7. *There exists a 2-regular* $E(18;9)$.

Proof. Base blocks B consist of

$$\{0_0, 0_0, 0_0\}, \{0_0, 4_1, 4_1\}, \{0_0, 2_0, 8_0\}, \{0_0, 4_0, 0_1\},$$

$$\{0_1, 3_1, 6_1\}, \{0_1, 1_1, 2_0\}, \{0_1, 2_1, 8_0\}, \{0_1, 4_1, 7_0\}.$$

Then (V, B) is a 2-regular $E(18; 9)$.

Lemma 2.8. *If $v \equiv 6$ (mod 12), then there exists a 2-regular $E(v; v/2)$.*

Proof. The case $v = 18$ is treated in Lemma 2.7. Let $v = 12t + 6$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$$B_1 \;:\; \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, 0_0\}\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (6t+2)_0\}\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

B_2 : a set of base blocks which form a cyclic $STS(6t+3)$ based on $Z_{6t+3} \times \{0\}$,

B_3 : $\{\{0_1, r_1, (b_r)_0\} | r = 1, 2, \cdots, 3t+1\}$

where $\{(a_r, b_r) | r = 1, 2, \cdots, 3t+1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$. Then (V, B) is a 2-regular $E(v; v/2)$.

Now, we have the following theorem.

Theorem 2.9. *There exists a 2-regular $E(v; v/2)$ if and only if $v \equiv 2$ or $6 (\mathrm{mod}\ 12)$.*

By Lemma 2.1, we know that if there exists a 3-regular $E(v; n)$, then $n = 0, v/3, 2v/3$ or v.

Lemma 2.10. *If there exists a 3-regular $E(v; n)$, then*
(i) $n = 0$ *and* $v \equiv 3 (\mathrm{mod} 6)$ *or*
(ii) $n = v/3$ *and* $v \equiv 9 (\mathrm{mod} 18)$ *or*
(iii) $n = 2v/3$ *and* $v \equiv 9 (\mathrm{mod} 18)$ *or*
(iv) $n = v$ *and* $v \equiv 3 (\mathrm{mod} 6)$.

Proof. (i) If $n = 0$, then $v \equiv 0 \pmod 3$ by Lemma 2.1. But we know that $v/3$ is an odd integer and hence $v \equiv 3 \pmod 6$.

(ii) If $n = v/3$, then $v/3 = 0 \pmod 3$ by Lemma 2.1; so $v \equiv 9 \pmod{18}$ since $v/3$ is odd.

(iii) If $n = 2v/3$, then $2v/3 \equiv 0 \pmod 3$ by Lemma 2.1; so $v \equiv 9 \pmod{18}$ because $v/3$ is odd.

(iv) If $n = v$, then the existence of 3-regular $E(v; v)$'s is equivalent to the existence of 3-regular $STS(v)$'s; so $v \equiv 3 \pmod 6$.

The following theorem is directly obtained from the spectrum for 3-regular STS's.

Theorem 2.11. *There exists a 3-regular $E(v;v)$ if and only if $v \equiv 3$ (mod 6).*

Lemma 2.12. *If $v \equiv 3$ (mod 6), then there exists a 3-regular $E(v;0)$.*

Proof. If $v = 9$, then a 3-regular $E(9;0)$ has base blocks

$$\{\{0_i, 0_i, 1_i\}, \{0_0, i_1, (2i)_2\} | i \in Z_3\}.$$

If $v \equiv 3$ (mod 6) and $v \neq 9$, let $V' = Z_v$ and let $\alpha' = (01 \cdots v-1)$. Base blocks: $B' = B_1 \cup B_2$ where

$B_1 : \{\{0, 0, v/3\}\}$,

B_2 : a set of base blocks which form a cyclic $STS(v)$, except the base block $\{0, v/3, 2v/3\}$, based on V'. Then (V', B') is a 3-regular $E(v;0)$ with $(\alpha')^3$ as a reguired automorphism.

Thus, we the following theorem.

Theorem 2.13. *There exists a 3-regular $E(v;0)$ if and only if $v \equiv 3$ (mod 3).*

Lemma 2.14. *There exists a 3-regular $E(27;9)$.*

Proof. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$$\begin{aligned}
B_1 &: \{0_0, 0_0, 0_0\}, \{0_0, 3_0, 6_0\}, \{0_0, 3_1, 6_2\}, \{0_0, 6_1, 3_2\}\},\\
B_2 &: \{0_i, 0_i, 3_i\} | i = 1, 2\},\\
B_3 &: \{\{0_i, 1_i, 2_{i+1}\}, \{0_i, 2_i, 7_{i+1}\}, \{0_i, 4_i, 8_{i+1}\} | i \in Z_3\}.
\end{aligned}$$

Then (V, B) is a 3-regular $E(27;9)$.

Lemma 2.15. *If $v \equiv 9$ (mod 18), then there exists a 3-regular $E(v;v/3)$.*

Proof. The case $v = 27$ is handled in Lemma 2.14, and let $v = 18t + 9$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

B_1 : $\{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (2t+1)_1\}, \{0_2, 0_2, (2t+1)_2\}\}$,

B_2 : a set of base blocks which form a cyclic $STS(6t+3)$ based on $Z_{6t+3} \times \{0\}$,

B_3 : a set of base blocks which form a cyclic $STS(6t+3)$, except the base blocks $\{0_i, (2t+1)_i, (4t+2)_i\}$, based on $Z_{6t+3} \times \{i\}$ for $i = 1, 2$

B_4 : $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+3}\}$. Then (V, B) is a 3-regular $E(v;v/3)$.

Now, we have

Theorem 2.16. *There exists a 3-regular* $E(v; v/3)$ *if and only if* $v \equiv 9(\bmod 18)$.

Lemma 2.17. *There exists a 3-regular* $E(27; 18)$.

Proof. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$B_1 : \{\{0_i, 0_i, 0_i\}, \{0_i, 3_i, 6_i\} | \, i = 0, 1\}$,

$B_2 : \{0_2, 0_2, 3_2\}, \{0_0, 3_1, 6_2\}, \{0_0, 6_1, 3_2\}\}$,

$B_3 : \{0_i, 1_i, 2_{i+1}\}, \{0_i, 2_i, 7_{i+1}\}, \{0_i, 4_i, 8_{i+1}\} | \, i \in Z_3\}$. Then (V, B) is a 3-regular $E(27; 18)$.

Lemma 2.18. *If* $v \equiv 9(\bmod 18)$, *then there exists a 3-regular* $E(v; 2v/3)$.

Proof. A 3-regular $E(27; 18)$ exists by Lemma 2.17, so let $v = 18t + 9$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

$B_1 : \{\{0_i, 0_i, 0_i\}, \{0_2, 0_2, (2t+1)_2\} | i = 0, 1\}$,

$B_2 : \{\{0_0, r_1, (2r)_2\} | r \in Z_{6y+3}\}$,

B_3 : a set of base blocks which form a cyclic $STS(6t+3)$ based on $Z_{6t+3} \times \{i\}$ for $i = 0, 1$

B_4 : a set of base blocks which form a cyclic $STS(6t+3)$, except the base block $\{0_2, (2t+1)_2, (4t+2)_2\}$, based on $Z_{6t+3} \times \{2\}$. Then (V, B) is a 3-regular.

Thus, we have

Theorem 2.19. *There exists a 3-regular* $E(v; 2v/3)$ *if and only if* $v \equiv 9(\bmod 18)$.

Let us construct 4-regular $E(v; n)$'s. First of all, by Lemma 2.1, if there exists a 4-regular $E(v; n)$ then $v = 0, v/4$ or $v/2$.

Lemma 2.20. *If there exists a 4-regular* $E(v; n)$, *then*

(i) $n = 0$ *and* $v \equiv 12(\bmod 24)$ *or*

(ii) $n = v/4$ *and* $v \equiv 4$ *or* $12(\bmod 24)$ *or*

(iii) $n = v/2$ *and* $v \equiv 12$ *or* $20(\bmod 24)$.

Proof. (i) If $n = 0$ then $v \equiv 0 \pmod 3$ by Lemma 2.1; so $v \equiv 12 \pmod{24}$ since $v/4$ is odd.

(ii) If $n = v/4$ then $v/4 \equiv 0$ or $1 \pmod 3$, equivalently $v \equiv 0$ or $4 \pmod{12}$; so $v \equiv 4$ or $12 \pmod{24}$ since $v/4$ is odd.

(iii) If $n = v/2$ then $v/2 \equiv 0$ or $1 \pmod 3$; so $v \equiv 12$ or $20 \pmod{24}$ since $v/4$ is odd.

Lemma 2.21. *There exists a 4-regular $E(36;0)$.*

Proof. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$B_1 : \{0_3, 0_3, 4_3\}, \{0_3, 2_3, 8_3\}, \{0_i, 0_i, 0_3\} \mid i = 0, 1, 2\}$,

$B_2 : \{\{0_0, r_1, (2r)_2\} \mid r \in Z_9\}$,

$B_3 : \{\{0_i, r_i, (b_r)_3 \mid i = 0, 1, 2; r = 1, 2, 3, 4\}$ where $\{(a_r, b_r) \mid r = 1, 2, 3, 4\}$ is an $(A, 4)$-system. Then (V, B) is a 4-regular $E(36; 0)$.

Lemma 2.22. *If $v \equiv 12 (\mathrm{mod} 24)$, then there exists a 4-regular $E(v;0)$.*

Proof. The case $v = 36$ is treated in Lemma 2.21. Let $v = 24t + 12$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

$$B_1 : \begin{cases} \{\{0_3, 0_3, (2t+1)_3\}, \{0_i, 0_i, 0_3\} \mid i = 0, 1, 2\} & \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{0_3, 0_3, (2t+1)_3\}, \{0_i, 0_i, (6t+2)_3\} \mid i = 0, 1, 2\} & \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

$B_2 : \{\{0_i, r_i, (b_r)_3\} \mid i = 0, 1, 2; r = 1, 2, \cdots, 3t+1\}$ where $\{(a_r, b_r) \mid r = 1, 2, \cdots, 3t+1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 \pmod 4$ or $t \equiv 2, 3 \pmod 4$.

$B_3 : \{\{0_0, r_1, (2r)_2\} \mid r \in Z_{6t+3}\}$,

B_4 : a set of base blocks which form a cyclic $STS(6t+3)$, except the base block $\{0_3, (2t+1)_3, (4t+2)_3\}$, based on $Z_{6t+3} \times \{3\}$. Then (V, B) is a 4-regular $E(v; 0)$.

Thus, we have

Theorem 2.23. *There exists a 4-regular $E(v;0)$ if and only if $v \equiv 12 (\mathrm{mod} 24)$.*

Lemma 2.24. *There exists a 4-regular $E(36;9)$.*

Proof. Base blocks: $B = B_1 \cup B_2 \cup B_3$ where

$$\begin{aligned} B_1 \ &: \ \{\{0_0, 0_0, 0_0\}, \{0_0, 2_0, 8_0\}, \{0_0, 4_0, 4_3\}, \{0_3, 1_3, 2_0\}, \\ &\quad\ \{0_3, 2_3, 8_0\}, \{0_3, 4_3, 7_0\}, \{0_3, 3_3, 6_3\}, \{0_3, 0_3, 4_0\}, \\ &\quad\ \{0_1, 0_1, 0_3\}, \{0_2, 0_2, 0_3\}\}, \\ B_2 \ &: \ \{\{0_0, r_1, (2r)_2\} \mid r \in Z_9\}, \\ B_3 \ &: \ \{\{0_i, r_i, (b_r)_3\} \mid i = 1, 2; r = 1, 2, 3, 4\} \end{aligned}$$

where $\{(a_r, b_r) \mid r = 1, 2, 3, 4\}$ is an $(A, 4)$-system. Then (V, B) is a 4-regular $E(36; 9)$.

Lemma 2.25. *If* $v \equiv 12 \pmod{24}$, *then there exists a 4-regular* $E(v; v/4)$.

Proof. A 4-regular $E(36; 9)$ exists by Lemma 2.24. Let $v = 24t + 12$ and $t \neq 1$. Base blocks: $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

$$B_1 : \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, 0_3\}, \{0_2, 0_2, 0_3\}, \{0_3, 0_3, 0_0\}\} \\ \text{if } t \equiv 0 \text{ or } 1 \pmod 4; \\ \{\{0_0, 0_0, 0_0\}, \{0_1, 0_1, (6t+2)_3\}, \{0_2, 0_2, (6t+2)_3\}, \{0_3, 0_3, (6t+2)_0\}\} \\ \text{if } t \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

B_2 : a set of base blocks which form a cyclic $STS(6t+3)$ based on $Z_{6t+3} \times \{0\}$,

B_3 : $\{\{0_1, r_1, (2r)_2\} | r \in Z_{6t+3}\}$,

B_4 : $\{\{0_1, r_1, (b_r)_3\}, \{0_2, r_2, (b_r)_3\}, \{0_3, r_3, (b_r)_0\} | r = 1, 2, \cdots, 3t+1\}\}$

where $\{(a_r, b_r) | r = 1, 2, \cdots, 3t+1\}$ is an $(A, 3t+1)$-system or a $(B, 3t+1)$-system depending on whether $t \equiv 0, 1 (\text{mod} 4)$ or $t \equiv 2, 3 (\text{mod} 4)$. Then (V, B) is a 4-regular $E(v; v/4)$.

Lemma 2.26. *If* $v \equiv 4 \pmod{24}$, *then there exists a 4-regular* $E(v; v/4)$.

Proof. Let $v = 24t + 4$. Base blocks: $B = B_1 \cup B_2 \cup B_3 \cup B_4$ where

$$B_1 : \begin{cases} \{\{0_0, 0_0, 0_0\}, \{0_3 0_3 0_0\}, \{0_i, 0_i, 0_3\} | i = 1, 2\} \\ \text{if } t \equiv 0 \text{ or } 3 \pmod 4; \\ \{\{0_0, 0_0, 0_0\}, \{0_3, 0_3, (6t)_0\}, \{0_i, 0_i, (6t)_3\} | i = 1, 2\} \\ \text{if } t \equiv 1 \text{ or } 2 \pmod 4, \end{cases}$$

B_2 : a set of base blocks which form a cyclic $STS(6t+1)$ based on $Z_{6t+1} \times \{0\}$,

B_3 : $\{\{0_0, r_1, (2r)_2\} | r \in Z_{6t+1}\}$,

B_4 : $\{\{0_3, r_3, (b_r)_0\}, \{0_i, r_i, (b_r)_3\} | i = 1, 2; r = 1, 2, \cdots, 3t\}$

where $\{(a_r, b_r) | r = 1, 2, \cdots, 3t\}$ is an $(A, 3t)$-system or a $(B, 3t)$-system depending on whether $t \equiv 0, 3 (\text{mod} 4)$ or $t \equiv 1, 2 (\text{mod} 4)$. Then (V, B) is a 4-regular $E(v; v/4)$.

Thus, we have

Theorem 2.27. *There exists a 4-regular* $E(v; v/4)$ *if and only if* $v \equiv 4$ *or 12* $(\text{mod} 24)$.

In the existence problem for 4-regular $E(v; n)$'s, the following case remains open: If $v \equiv 12$ or $20 (\text{mod} 24)$, does there exist a 4-regular $E(v; v/2)$?

References

[1] Bennett, F.E., Mendelsohn, N.S., *On the existence of extended triple systems*, Utilitas Math., 14(1978), 249-267.

[2] Cho, C.J., *Rotational extended triple systems*, to appear.

[3] Johnson, D.M., Mendelsohn, N.S., *Extended triple systems*, Aequationes Math., 3(1972), 291-298.

[4] O'Keefe, E.S., *Verification of a conjecture of Th. Skolem*, Math. Scand., 9(1961), 80-82.

[5] Peltesohn, R., *Eine Lösung der beiden heffterschen Differenzenprobleme*, Compositio Math., 6(1939), 251-257.

[6] Rosa, A., *Poznamka o cyklickych Steinerovych systemoch trojic*, Math. Fyz. Cas., 16(1966), 285-290.

[7] Skolem, Th., *On certain distributions of integers in pairs with given differences*, Math. Scand., 5(1957), 57-68.

DEPARTMENT OF MATHEMATICS, SOOKMYUNG WOMEN'S UNIVERSITY, SEOUL, 140-742, KOREA.