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ON THE LIMIT-POINT CLASSIFICATION OF A WEIGHTED FOURTH ORDER DIFFERENTIAL EXPRESSIONS

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Conditions are given to ensure that a weighted fourth order differential equation on a half-line is in the case of the limit-point at infinity. These conditions are in the form of limitations on the growth of these coefficients.

1. Introduction

The differential expression considered in this paper is:

$$M[y] \equiv (P_0(x)y^{(2)})^{(2)} - (P_1(x)y')' + P_2(x)y = \lambda h(x)y \qquad (1.1)$$

where λ is a complex parameter, will be regular at all points of $[0, \infty]$ but has a singular point at infinity, see (3).

It is assumed the coefficients $P_0(x)$, $P_1(x)$ and $P_2(x)$ satisfy the following basic conditions;

i- $P_2(x)$ is locally Lebesgue-integrable on $[0, X]$, for all $X > 0$.

ii-$P_0'(x)$ and $P_1(x)$ are absolutely continuous on $[0, X]$, for all $X > 0$.

iii-$P_0(x) > 0$ and $P_1(x) \geq 0$, for all $x \in [0, \infty]$. The weight function $h(x)$ satisfies.

iv-$h'(x)$ is continuous on $[0, \infty]$ and $h(x) > 0$.

We introduce the Hilbert space $L_h^2[a, \infty)$ of complex valued measurable functions $f(x)$ such that:

$$\int_a^\infty |f(x)|^2 h(x)dx$$

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converges and the inner product of $f(x)$ and $g(x)$) in $L_h^2[a,\infty)$ is defined by

$$(f(x), g(x) = \int_a^\infty f(x)\overline{g(x)}h(x)dx$$

Classical results give that the number m of linearly independent $L_h^2[a,\infty)$ solutions of $M(y) = \lambda hy$ is the same for all nonreal λ and satisfy the inequality $2 \leq m \leq 4$.

When $m = 2$, the operator is said to be in the limit-point case at infinity.

2. System of Differential Equations

Through this section, we assume ρ_0, ρ_1 are positive functions on $[a,\infty)$, with second continuous derivatives and λ denotes a complex number with $Re\lambda = 0$, we consider the conditions:

$$\frac{P_1(x)\rho_1^2}{P_0(x)\rho_0^2 h^{\frac{1}{2}}} = 0(1); \frac{\rho_1}{\rho_0 h^{\frac{1}{4}}}[\frac{\rho_0'}{\rho_0} + \frac{\rho_1'}{\rho_1} + \frac{P_0'}{P_0} + \frac{h'}{h}] = 0(1) \text{ as } x \to \infty \tag{2.1}$$

and

$$\frac{-P_2(x)\rho_1^4}{P_0(x)\rho_0^4 h} \leq k, \text{ for some } K > 0 \tag{2.2}$$

For (1.1), the quasi-derivatives $y^{[i]}$ are defined by:

$$\begin{gathered} y^{[0]} = y; y^{[1]} = y', \quad y^{[2]} = P_0(x)y'' \\ y^{[3]} = P_1(x)y' - (P_0 y'')' \text{ and } y^{[4]} = \lambda hy \end{gathered} \tag{2.3}$$

The equation $M(y) = \lambda hy$, has the vector formulation:

$$Y' = AY \tag{2.4}$$

where

$$Y = \begin{bmatrix} y \\ y' \\ y^{[2]} \\ y^{[3]} \end{bmatrix} \quad \text{and } A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1/P_0 & 0 \\ 0 & P_1 & 0 & -1 \\ (P_2 - \lambda h) & 0 & 0 & 0 \end{bmatrix}$$

We transform y by the transformation $W = LY$, where L is the diagonal matrix.

$$L = \text{ diagonal } \{\rho_0 h^{\frac{3}{8}}, \rho_1 h^{\frac{1}{8}}, (\rho_1^2/\rho_0 P_0 h^{\frac{1}{8}}), (\rho_1^3/\rho_0^2 P_0 h^{\frac{3}{8}})\}$$

Clearly, the vector $W = [w_1, w_2, w_3, w_4]^T$ is;

$$W = [(\rho_0 h^{\frac{3}{8}} y, (\rho_1 h^{\frac{1}{8}})y', (\rho_1^2/\rho_0 P_0 h^{\frac{1}{8}})y^{[2]}, (\rho_1^3/\rho_0^2 P_0 h^{\frac{3}{8}})y^{[3]}]^T \tag{2.5}$$

The vector W satisfies:

$$W' = (\rho_0 h^{\frac{1}{4}}/\rho_1)CW, \tag{2.6}$$

where

$$C = (\rho_1/\rho_0 h^{\frac{1}{4}})[LAL^{-1} + L'L^{-1}].$$

Calculations show that $C = \{C_{ij}\}$ satisfies; $C_{i,i+1} = \pm 1, C_{ii}$ is bounded $(i = 1, 2, 3, 4)$ (by (2.1)),

$$\begin{aligned} C_{32} &= \frac{P_1 \rho_1^2}{P_0 \rho_0^2 h^{\frac{1}{2}}} = 0(1) \text{ as } x \to \infty \\ C_{44} &= (P_2 - \lambda h)(\rho_1^4/\rho_0^4 P_0 h) \geq -k \text{ (by (2.2)),and otherwise} \\ C_{ij} &= 0 \end{aligned}$$

We take, for $k = 1, 2, 3, 4$.

$$\sqrt{\frac{\rho_0 h^{\frac{1}{4}}}{\rho_1}} w_k = \left(\frac{\rho_0^{5-2k} h^{\frac{3-k}{2}}}{\rho_1^{3-2k}}\right)^{\frac{1}{2}} \frac{1}{P_0^{\alpha k}} y^{[k-1]}$$

where

$$\alpha_k = \delta_{3k} + \delta_{4k} \text{ (kronical delta).}$$

Thus, if we define

$$W_k = \int_a^t |w_k|^2 \frac{\rho_0 h^{\frac{1}{4}}}{\rho_1} ds \tag{2.7}$$

3. Auxiliary lemmas

In this section we state lemmas from (1), and the proof of the results of this paper depends on the following lemma;

Lemma 3.1. *If W is a solution of (2.6) such that C_{ij} are bounded for all i and j and if (2.7) holds for $k = 1, 2, 3, 4$, such that $W_1(\infty) < \infty$. Then, W_i and w_i^2 are $0(1)$ for $i = 1, 2, 3, 4$, more-over,*

$$\begin{aligned} W_i &= 0(W_{i+1}^{1-\frac{1}{i}}); i = 1, 2, 3, \text{ as } x \to \infty \\ w_i^2 &= 0(W_{i+1}^{1-\frac{1}{2i}}); i = 1, 2, 3 \text{ as } x \to \infty \end{aligned}$$

Proof. Since $W_1(\infty) < \infty$, $W_1(x) = 0(1)$ as $x \to \infty$, we rewrite

$$W_1 = 0(W_2^{\frac{0}{1}}), \text{ as } x \to \infty$$

For w_k, write generally that:

$$w_k^2|_a^t = 2\int_a^t w_k w_k' ds \tag{3.1}$$

Thus, for $k = 1$, by (2.5) and (2.6), we get:

$$\begin{aligned} w_1^2|_a^t &= 2\int_a^t w_1 w_1' ds = 2\int_a^t (\rho_0 h^{\frac{3}{8}}) y[(\rho_0 h^{\frac{3}{8}})' y + \rho_0 h^{\frac{3}{8}} y^{[1]}] ds \\ &= 2\int_a^t \frac{\rho_0(\rho_0 h^{\frac{3}{8}})'}{h^{\frac{5}{8}}} \frac{\rho_0^3 h}{\rho_1} |y|^2 ds + 2\int_a^t \rho_0^2 h^{\frac{3}{4}} y y^{[1]} ds \end{aligned}$$

The first integral of the right hand-side is $0(w_1)$ $[\frac{\rho_0(\rho_0 h^{\frac{3}{8}})'}{h^{\frac{5}{8}}}] = 0(1)$ as $x \to \infty$, and by using Cauchy Schwartz inequality, the second integral is:

$$\begin{aligned} \int_a^t \rho_0^2 h^{\frac{3}{4}} y y' ds &= 0([\int_a^t \frac{\rho_0^3 h |y|^2}{\rho_1} ds]^{\frac{1}{2}} \cdot [\int_a^t \rho_0 \rho_1 h^{\frac{1}{2}} |y'|^2 ds]^{\frac{1}{2}}) \\ &= 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) \text{ as } t \to \infty [\text{by (2.7)}] \end{aligned}$$

Thus, we can write that: $w_1^2|_a^t = 0(W_1^{\frac{1}{2}}[W_1^{\frac{1}{2}} + W_2^{\frac{1}{2}}])$ as $x \to \infty$, Since $W_1 = 0(1)$ as $t \to \infty$. Then,

$$w_1^2|_a^t = 0(W_1^{\frac{1}{2}}) \text{ as } t \to \infty \tag{3.2}$$

Next, for $k = 2$, in (2.7), we get:

$$W_2 = \int_a^t \frac{\rho_0 h^{\frac{1}{4}}}{\rho_1} |w_2^2| ds. \tag{3.3}$$

Now, by (2.6):

$$W_2 = \int_a^t w_2 w_1' ds - \int_a^t \frac{\rho_0^2 h^{\frac{5}{8}}}{\rho_1} \frac{(\rho_0 h^{\frac{3}{8}})' y \rho_1}{\rho_0^2 h^{\frac{5}{8}}} (\rho_1 h^{\frac{1}{8}}) y^{[1]} ds$$

Integrating by parts the first integral of right hand side, we get;

$$W_2 = w_2 w_1|_a^t - \int_a^t w_1 w_2' ds - \int_a^t \frac{\rho_0^2 h^{\frac{5}{8}}}{\rho_1} y \cdot \frac{(\rho_0 h^{\frac{3}{8}})' \rho_1}{\rho_0^2 h^{\frac{5}{8}}} (\rho_1 h^{\frac{1}{8}}) y' ds$$

From (2.7) and (2.4) we get:

$$\begin{aligned}
W_2 &= w_2 w_1|_a^t - \int_a^t [\frac{(\rho_1 h^{\frac{1}{8}})'}{\rho_0 h^{\frac{3}{8}}}(\rho_0 h^{\frac{3}{8}})y' + \frac{\rho_1 h^{\frac{1}{8}} y^{[2]}}{P_0}](\rho_0 h^{\frac{3}{8}})y ds \\
&- \int_a^t \frac{\rho_0^2 h^{\frac{5}{8}}}{\rho_1} y \frac{(\rho_0 h^{\frac{3}{8}})'\rho_1}{\rho_0^2 h^{\frac{5}{8}}}(\rho_1 h^{\frac{1}{8}})y' ds \\
&= w_2 w_1|_a^t - \int_a^t \frac{(\rho_1 h^{\frac{1}{8}})'}{\rho_0 h^{\frac{3}{8}}}(\rho_0^2 h^{\frac{3}{4}})yy' ds - \int_a^t \frac{\rho_0\rho_1 h^{\frac{1}{2}}}{P_0} yy^{[2]} ds \\
&- \int_a^t \frac{(\rho_0 h^{\frac{3}{8}})'\rho_1}{\rho_0^2 h^{\frac{5}{8}}}(\rho_0^2 h^{\frac{3}{4}})yy' ds.
\end{aligned}$$

By using Cauchy Schwartz inequality and from (2.1) and (2.7), we get,

$$\begin{aligned}
W_2 &= w_1 w_2 + 0([\int_a^t \frac{\rho_0 h|y|^2}{\rho_1} ds]^{\frac{1}{2}}[\int_a^t \rho_0\rho_1 h^{\frac{1}{2}}|y'|^2 ds]^{\frac{1}{2}} \\
&+[\int_a^t \frac{\rho_0^3 h}{\rho_1}|y|^2 ds]^{\frac{1}{2}}[\int_a^t \frac{\rho_1^3}{\rho_0 P_0}|y^{[2]}|^2 ds]^{\frac{1}{2}} \\
&+[\int_a^t \frac{\rho_0^3 h}{\rho_1}|y|^2 ds]^{\frac{1}{2}}[\int_a^t \rho_0\rho_1 h^{\frac{1}{2}}|y'|^2 ds]^{\frac{1}{2}}
\end{aligned}$$

Thus, we can write:

$$W_2 = w_1 w_2 + 0(W_1^{\frac{1}{2}} W_2^{\frac{1}{2}} + W_1^{\frac{1}{2}} W_3^{\frac{1}{2}} + W_1^{\frac{1}{2}} W_2^{\frac{1}{2}}) \text{ as } t \to \infty.$$

Since $W_1(x) = 0(1)$ as $x \to \infty$, Then; By (3.1), for $k = 2$,

$$w_2^2|_a^t = 2\int_a^t w_2 w_2' ds = 2\int_a^t \frac{(\rho_1 h^{\frac{1}{8}})'}{\rho_0 h^{\frac{3}{8}}}\rho_1\rho_0 h^{\frac{1}{2}}|y'|^2 ds + \int_a^t \frac{\rho_1^2 h^{\frac{1}{4}} y' y^{[2]}}{P_0} ds$$

By using (2.1) the first integral of the right hand side is $0(W_2)$ as $t \to \infty$ and by using Cauchy-Schwartz inequality the second integral is $0(W_2^{\frac{1}{2}} W_3^{\frac{1}{2}})$ as $t \to \infty$. Thus,

$$w_2^2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]) \text{ as } t \to \infty. \tag{3.4}$$

While $w_1 = 0(W_2^{\frac{1}{4}})$ as $t \to \infty$, hence

$$w_1 w_2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]^{\frac{1}{2}}) \text{ as } t \to \infty \tag{3.5}$$

From (3.3) and (3.5), we get:

$$W_2 = 0(W_2^{\frac{1}{2}}[W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}}]^{\frac{1}{2}}) + 0(W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}})$$

Thus, after division by W_2, we get: $1 = 0(W_2^{-\frac{1}{2}} + I)$ as $t \to \infty$, where $I(t) = \frac{W_3^{\frac{1}{2}}}{W_2}$, hence

$$\liminf_{t\to\infty} I(t) > 0, \text{ i.e. } W_2 = 0(W_3^{\frac{1}{2}}) \text{ as } t \to \infty \tag{3.6}$$

Further return to (3.4) and from (3.6) we get:

$$w_2^2 = 0(W_3^{\frac{3}{4}}) \text{ as } t \to \infty$$

i.e.

$$w_2 = 0(W_3^{\frac{3}{8}}) \text{ as } t \to \infty.$$

In a similar way, one can prove that at $t \to \infty$

$$W_3 = 0(W_4^{\frac{2}{3}}) \text{ and } w_3 = 0(W_4^{\frac{5}{12}})$$

Finally

$$W_4 = 0(1) \text{ as } t \to \infty \text{ and } w_4 = 0(1) \text{ as } t \to \infty.$$

This completes the proof.

The Lagrange identity for $h^{-1}M[y]$ is:

$$\int_a^b (h^{-1}Mf)h\bar{g}ds - \int_a^b hf\overline{(h^{-1}Mg)ds} = [f,g]'$$

where

$$[f,g] = [f\bar{g}^{[3]} - f^{[3]}\bar{g}] + [f'\bar{g}^{[2]} - f^{[2]}\bar{g}'] \tag{3.7}$$

We note that: If $M[y] = \lambda hy$ and $M[z] = \bar{\lambda}h\bar{z}$ implies $[y,z]' = 0$ [See (1)]. For $M[y] = \lambda hy$, the quadratic expression is:

$$[y^{[3]}\bar{y} + y^{[2]}\bar{y}']' = \frac{1}{P_0}|y^{[2]}|^2 + P_1|y'|^2 + (P_2 - \lambda h)|y|^2. \tag{3.8}$$

We also make use of the vector spaces.

$$\begin{aligned} V &= \{f : Mf = \lambda hf\} \\ V_+ &= \{y : My = \lambda hy\} \cap L_h^2[a,\infty) \\ V_- &= \{z : Mz = \bar{\lambda}h\bar{z}\} \cap L_h^2[a,\infty). \end{aligned}$$

Lemma 3.2. *If* $\dim(V_+) + \dim(V_-) > 4$, *then there is a* $y \in V_+$ *and* $z \in V_-$ *such that* $[y, z] = 1$.

For the proof see (1).

Lemma 3.3. *Let* F *be a non-negative continuous function on* $[a, \infty)$, *and define*

$$H(t) = \int_a^t (t-s)^2 F(s) ds.$$

If as $t \to \infty; H(t) = 0(t^2[H'']^{\frac{3}{4}})$, *then*

$$\int_a^t F(S) ds = 0(1) \text{ as } t \to \infty.$$

For the proof see (1).

Next define

$$\begin{aligned} J_1 &= W_3(t) = \int_a^t \frac{\rho_1^3 |y^{[2]}|^2}{\rho_0 P_0^2} ds, y \in V_+ \\ J_2 &= W_3(t) = \int_a^t \frac{\rho_1^3 |z^{[2]}|^2}{\rho_0 P_0^2} ds, z \in V_- \end{aligned}$$

$J_1(\infty)$ and $J_2(\infty)$ are $0(1)$ by lemma (3.1).

Theorem 3.1. *If there are two positive continuous functions* ρ_0 *and* ρ_1 *such that:*

$$\frac{P_1 \rho_1^2}{P_0 \rho_0^2 h^{\frac{1}{2}}}, \frac{\rho_1}{\rho_0 h^{\frac{1}{4}}}[\frac{\rho_0'}{\rho_0} + \frac{\rho_1'}{\rho_1} + \frac{P_0'}{P_0} + \frac{h'}{h}] \text{ are } 0(1) \text{ as } x \to \infty \tag{3.9}$$

$$\frac{-P_2 \rho_1^4}{P_0 \rho_0^4 h} \leq k, \text{ for same } k > 0 \tag{3.10}$$

and

$$\int_a^\infty \frac{h^{\frac{1}{4}} \rho_1^2}{P_0} ds = \infty \tag{3.11}$$

Then, $h^{-1} M[y]$ *is the limit-point case at infinity.*

Proof. We first show $J_1(\infty) < \infty$, from (3.8) we have:

$$\begin{aligned} I &= \int_a^t \{\frac{1}{P_0}|y^{[2]}|^2 + P_1|y'|^2 + (P_2 - \lambda h)|y|^2\}(1 - \frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0} ds \\ &= \int_a^t \{y^{[3]}\bar{y} + y^{[2]}\bar{y}'\}(1 - \frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0} ds. \end{aligned} \tag{3.12}$$

Integrating by parts the left hand side, we get:

$$I = 0(1) - \int_a^t [y^{[2]}y' + y^{[3]}y]\{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}' ds.$$

By using the concepts of (2.3) we obtain:

$$I = 0(1) - \int_a^t \{P_1 y' y - (y^{[2]})' y + y^{[2]} y'\}[(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}]' ds \tag{3.13}$$

The last factor of the above integration can be estimated as follows:

$$[(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}]' = [k_1 \frac{\rho_1}{t\rho_0 h^{\frac{1}{4}}} + k_2 \frac{\rho_1'}{\rho_0 h^{\frac{1}{4}}} + k_3 \frac{\rho_0' \rho_1}{\rho_0^2 h^{\frac{1}{4}}} + k_4 \frac{\rho_1 P_0'}{\rho_0 P_0 h^{\frac{1}{4}}}] \frac{\rho_1^2 h^{\frac{1}{4}}}{P_0},$$

where $k_j, J = 1,2,3,4$ are constants by (3.9) we get:

$$\{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}' = 0(\frac{\rho_1^2 h^{\frac{1}{4}}}{P_0}) \text{ as } t \to \infty.$$

Hence

$$\begin{aligned}
&\int_a^t (P_1 y' y + P_0 y'' y')\{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}' ds \\
&= 0(\int_a^t (\rho_0\rho_1)^{\frac{1}{2}} h^{\frac{1}{4}} y' \cdot (\frac{\rho_0^3 h}{\rho_1})^{\frac{1}{2}} y \cdot (\frac{P_1 \rho_1^2 h^{\frac{1}{4}}}{P_0 \rho_0^2 h^{\frac{3}{4}}}) ds \\
&\quad + \int_a^t \frac{P_0 \rho_1^2 h^{\frac{1}{4}}}{P_0 \rho_1^2 h^{\frac{1}{4}}} \cdot (\frac{\rho_1^3}{\rho_0})^{\frac{1}{2}} y'' \cdot (\rho_0 \rho_1 h^{\frac{1}{2}})^{\frac{1}{2}} y' ds) \\
&= 0(W_2^{\frac{1}{2}} W_1^{\frac{1}{2}} + W_2^{\frac{1}{2}} W_3^{\frac{1}{2}}) \text{ as } t \to \infty
\end{aligned} \tag{3.14}$$

Using Cauchy-Schwartz inequality and by (3.9), we get:

$$\begin{aligned}
&\int_a^t (P_0 y'')' y \{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}' = 0(1) \\
&- \int_a^t P_0 y'' \cdot y' \{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}' ds - \int_a^t P_0 y'' y \{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}'' ds
\end{aligned}$$

The last factor of the second integral in the right hand side can be estimated as follows:

$$\{(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}\}'' = 0(\frac{\rho_1^2 h^{\frac{1}{4}}}{P_0})' = 0(\frac{\rho_0 \rho_1 h^{\frac{1}{2}}}{P_0}) \text{ as } t \to \infty.$$

Hence

$$\begin{aligned}\int_a^t (P_0 y'')' y\{(1-\frac{s}{t})^2 \frac{\rho_1^3}{P_0\rho_0}\}' &= 0(\int_a^t \frac{P_0\rho_1^2 h^{\frac{1}{4}}}{P_0\rho_1^2 h^{\frac{1}{4}}}(\frac{\rho_1^3}{\rho_0})^{\frac{1}{2}} y'' \cdot (\rho_0\rho_1 h^{\frac{1}{2}})^{\frac{1}{2}} y' ds \\ &\quad + \int_a^t (\frac{\rho_1^3}{\rho_0})^{\frac{1}{2}} y'' \cdot (\frac{\rho_0^3 h}{\rho_1})^{\frac{1}{2}} y \cdot \frac{P_0\rho_0\rho_1 h^{\frac{1}{2}}}{P_0\rho_0\rho_1 h^{\frac{1}{2}}} ds) \\ &= 0(W_3^{\frac{1}{2}} W_2^{\frac{1}{2}} + W_3^{\frac{1}{2}} W_1^{\frac{1}{2}}) \text{ as } t \to \infty \quad (3.15)\end{aligned}$$

From (3.13); (3.14) and (3.15), we have:

$$\begin{aligned}\int_a^t \{y^{[3]} y + y^{[2]} y'\}'(1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0}) ds &= 0(W_3^{\frac{1}{2}} W_2^{\frac{1}{2}}) \\ &= 0(J_1^{\frac{1}{2}} J_1^{\frac{1}{4}}) = 0(J_1^{\frac{3}{4}}) \text{ as } t \to \infty\end{aligned}$$

Next,

$$\begin{aligned}\int_a^t P_1 y'^2 (1-\frac{s}{t})^2 \frac{\rho_1^3}{P_0\rho_0} ds &= 0(\int_a^t \rho_0\rho_1 h^{\frac{1}{2}} |y'|^2 \frac{P_1\rho_1^2}{P_0\rho_0^2 h^{\frac{1}{2}}} ds \\ &= 0(W_2) = 0(J_1^{\frac{1}{2}}) \text{ as } t \to \infty\end{aligned}$$

Also,

$$\begin{aligned}\int_a^t (P_2 - \lambda h)|y|^2 (1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0} ds &= 0(\int_a^t \frac{\rho_0^3 h}{\rho_1} |y|^2 ds) \\ &= 0(W_1) \text{ as } t \to \infty\end{aligned}$$

These inequalities in (3.12) give:

$$\int_a^t \frac{1}{P_0} |y^{[2]}|^2 (1-\frac{s}{t})^2 \frac{\rho_1^3}{\rho_0 P_0} ds = 0(t^2 J_1^{\frac{3}{4}}) \text{ as } t \to \infty$$

[by lemma (3.3)] with $F = \frac{\rho_1^3 |y^{[2]}|^2}{\rho_0 P_0^2}$, now applies to yield $J_1(\infty) < \infty$. Similarly, $J_2(\infty) < \infty$.

If $Im\lambda = 0$, then $V_+ = V_-$. If $Im\lambda \neq 0$, there is one to one correspondence from V_+ onto V_-, i.e. $\dim V_+ = \dim V_-$.

If $\dim V_+ > 2$ then $\dim V_+ + \dim V_- > 4$, and by lemma (3.2), then $[y, z] \equiv 1$, for $y \in V_+$ and $z \in V_-$, hence from (3.7), we have;

$$(y' z^{[2]} - y^{[2]} z') + (y z^{[3]} - y^{[3]} z) \equiv 1.$$

Hence

$$\int_a^t (y'z^{[2]} - y^{[2]}z')(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds + \int_a^t (yz^{[3]} - y^{[3]}z)\frac{h^{\frac{1}{4}}\rho_1^2(1-\frac{s}{t})}{P_0}ds$$
$$= \int_a^t (1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds \tag{3.16}$$

Now, the first integral on the left hand side can be estimated as follows: Since

$$(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0} = 0(\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}) \text{ as } t \to \infty.$$

Hence

$$\begin{aligned} y'z^{[2]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0} &= 0(y'z^{[2]}\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}) \text{ as } t \to \infty \\ &= 0([\frac{\rho_1^3|z^{[2]}|^2}{\rho_0 P_0}]^{\frac{1}{2}} \cdot [\rho_0\rho_1 h^{\frac{1}{2}}|y'|^2]^{\frac{1}{2}} \cdot \frac{h^{\frac{1}{4}}\rho_1^2}{P_0}\frac{P_0}{h^{\frac{1}{4}}\rho_1^2}) \\ &= 0([\frac{\rho_1^3|z^{[2]}|^2}{\rho_0 P_0}]^{\frac{1}{2}}[\rho_0\rho_1 h^{\frac{1}{2}}|y'|^2]^{\frac{1}{2}} \text{ as } t \to \infty \end{aligned}$$

This means that:

$$\int_a^t y'z^{[2]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds = 0([\int_a^t \frac{\rho_1^3|z^{[2]}|^2}{P_0\rho_0}ds)^{\frac{1}{2}} \cdot [\int_a^t \rho_0\rho_1 h^{\frac{1}{2}}|y'|^2 ds]^{\frac{1}{2}}$$

By Cauchy-Schwartz inequality, we get:

$$\begin{aligned} \int_a^t y'z^{[2]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds &= 0(W_3^{\frac{1}{2}}W_2^{\frac{1}{2}}) \text{ as } t \to \infty \\ &= 0(J_2^{\frac{1}{2}}J_1^{\frac{1}{4}}) = 0([J_1J_2]^{\frac{1}{2}}) \text{ as } t \to \infty \end{aligned}$$

Also, the integral

$$\int_a^t z'y^{[2]}(1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds = 0(J_1^{\frac{1}{2}}J_2^{\frac{1}{4}}) = 0([J_1J_2]^{\frac{1}{2}}) \text{ as } t \to \infty$$

All these inequalities, in (3.16) give:

$$\limsup_{t\to\infty} \int_a^t (1-\frac{s}{t})\frac{h^{\frac{1}{4}}\rho_1^2}{P_0}ds < \infty$$

and this contrary to the condition (3.11) of the present theorem. This means that $\dim V_+ = 2$, and the proof is complete.

4. General Application

If $P_0(t) = t^{\alpha_0}$, $P_1(t) = t^{\alpha_1}$, $P_2(t) = t^{\alpha_2}$ and $h(t) = t^{\alpha_3}$, where

$$\alpha_0 \leq \alpha_3 + 4; \alpha_k = \frac{(3-2k)\alpha_0 + 2k + 2k\alpha_3}{3}; k = 1, 2.$$

Then, $h^{-1}M[y]$ is in the case of limit-point at infinity.

Proof:

It may be verified that conditions (3.9) $\to$ (3.11) hold with

$$\rho_0 = t^{(\alpha_0/6 - \frac{1}{6} - \frac{\alpha_3}{24})}$$

and

$$\rho_1 = t^{(\frac{\alpha_0}{2} - \frac{1}{2} - \frac{\alpha_3}{8})}.$$

As special choice of the above theorem, let $\alpha_3 = 0$, thus

$$3\alpha_1 = \alpha_0 + 2, 3\alpha_2 = -\alpha_0 + 4 \text{ and } \alpha_0 \leq 4. \tag{4.1}$$

We notice that conditions (4.1) are more general than those of (2). Here if we take $\alpha_0 = 0$, we get the equation mentioned in (2) which is

$$(\psi^{(2)})^{(2)} - (P_1\psi')' + P_2\psi = \lambda\psi.$$

Then

$$P_1 = k_1 t^{\frac{2}{3}} \text{ and } P_2 = k_2 t^{\frac{4}{3}}$$

which is Everitt result (2).

Hinton (1) has given the following criterion for the equation

$$M[y] = \sum_{k=0}^{n} (-1)^k (t^{\alpha_{n-k}} y^{(k)})^{(k)}$$

where,

$$\alpha_0 \leq 2n, \quad \alpha_{n-k} = \frac{4(n-k) + \alpha_0(4k-2)}{4n-2} \text{ and}$$

$$\alpha_n = \frac{4n - 2\alpha_0}{4n-2}.$$

If $n = 2$, in this result we get:

$$\alpha_0 \leq 4, 3\alpha_1 = \alpha_0 + 2, \text{ and } 3\alpha_2 = -\alpha_0 + 4$$

which is our special result (4.1).

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