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ON ASYMPTOTIC ANALYSIS OF SINGULAR DIFFERENTIAL SYSTEMS

A. Zaghrout

The singular initial value problem $dY/dt = \mu F(t, Y, \mu)$, $Y(0, \mu) = Y_0$, is considered. A deductive asymptotic analysis is developed. With the aid of a local average value of the solution Y, asymptotic approximations of Y are investigated. The main tool in our analysis is the method of Krilov-Gogolioubov-Mitropolski.

1. Introduction

In this section we consider the singular initial value problem

$$\frac{dY}{dt} = \mu F(t, Y, \mu), \quad Y(0, \mu) = Y_0 \tag{1.1}$$

where $F : R \times R^n \times R \to R^n$ is defined in some connected subset $G \subset R^{n+2}$, and F has the components

$$f_i(t, y_1, y_2, y_3, \cdots, y_n, \mu).$$

Our main hypotheses on F, to ensure the existence, uniqueness and continuation of the solution $Y(t, \mu)$ of (1.1), are:

H_1) F is a continuous and uniformly bounded function in G, where $G = [Y : Y \in \overline{D}] \times I_t \times I_\mu$, D is an open bounded subset of R^n $\lim_{\mu \to 0}[F(t, Y, \mu) - F(t, Y, 0)] = 0$ uniformly in $(Y : Y \in \overline{D}) \times I_t$ where

$$I_t = \{t : 0 \leq t, \infty\}, \quad I_\mu = \{\mu : 0 \leq \mu < \mu_0\}.$$

H_2) F satisfies a Lipschitz condition with respect to Y in G i.e.

$$|F(t, Y_1, \mu) - F(t, Y_2, \mu)| \leq \lambda |Y_1 - Y_2|$$

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where $Y_1, Y_2 \in D$ and λ is a constant.

The existence, uniqueness and continuation of the solution $Y(t,\mu)$ can be obtained by the following two classical lemmas (for the proof see Roseau [6]).

Lemma A. *If F satisfies the two hypotheses H_1 and H_2, then there exists a unique solution $Y(t,\mu)$ of the system (1.1) in $0 \le t \le T$, $Y_0 \in D$ with $T = \frac{d}{\mu M}$ where d is the distance of Y_0 to the boundary of D and M is defined by*

$$M = \sup_G |F|.$$

According to Lemma A, it is meaningful to study $Y(t,\mu)$ on the natural time scale μ^{-1}.

Let $\tau = t$ and $Y^*(\tau,\mu) = Y(\frac{\tau}{\mu},\mu)$, then we have the system

$$\frac{dY^*}{d\tau} = F(\frac{\tau}{\mu}, Y^*, \mu), \quad Y^*(0,\mu) = Y_0. \tag{1.2}$$

Lemma B. *Let I be a closed interval such that, for $\tau \in I$, the unique solution $Y^*(\tau,\mu)$ of the system (1.2) exists and $Y^*(\tau,\mu) \in K$ where K is a compact subset of D. Then a unique continuation of $Y^*(\tau,\mu)$ exists in some open interval containing I. Furthermore, the solution $Y^*(\tau,\mu)$ may be continued to all values of τ, for which the continuation remains in a compact subset of D.*

It is well known that the initial value problem (1.2) is equivalent to the integral equation

$$Y(\tau,\mu) = Y_0 + \int_0^\tau F(\frac{r}{\mu}, Y(r,\mu), \mu)dr. \tag{1.3}$$

According to the asymptotic method (Krilov-Bogolioubov-Mitropolski [1]), $Y(t,\mu)$ can be approximated by the function $Z(\mu t)$ defined as a solution of

$$\frac{dZ}{dt} = \mu F_0(Z), \quad Z(0) = Y_0$$

where $F_0(Z) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,z,0)dt$.

The approximation is valid in the sense that

$$|Y(t,\mu) - Z(\mu t)| \to 0 \text{ as } \mu \to 0.$$

The validity is assured in an interval $0 \leq t \leq \frac{L}{\mu}$ where L is a constant (independent of μ).

We shall develop a deductive asymptotic analysis. A local average value of the function Y is defined. With the aid of this concept a deductive procedure establishes the fundamental of Krilov-Bogolioubov-Mitropolski under more general conditions.

We shall investigate asymptotic approximations of $Y(t, \mu)$ valid for $\mu \to 0$. The definitions of asymptotic approximations are closely related to the definitions of orders of magnitude of functions. Let $\Phi : R \times R \to R^n$ be any vector function with components $\phi_1, \phi_2, \phi_3, \cdots, \phi_n$. We define

$$|\Phi(t, \mu)| = \sum_{i=1}^{n} |\phi_i(t, \mu)|$$

and

$$\|\Phi\|_J = \max_{t \in J} |\Phi(t, \mu)|,$$

J denotes a bounded closed interval (which may depend on μ) and such Φ is continuous on J, $\delta(u)$ will denote a real positive continuous function of μ with the property that

$$\lim_{\mu \to 0} \delta(\mu) \text{ exists .}$$

Such functions are called order functions.

Definition 1. $\Phi(t, \mu) = O(\delta)$ in J if there exists a constant K such that $\|\Phi\|_J \leq K\delta(\mu)$, $\mu \in I_\mu$.

$$\Phi(t, \mu) = O(J) \text{ in } J \text{ if } \lim_{\mu \to 0} \frac{\|\Phi\|_J}{\delta} = 0.$$

Definition 2. δ_s^{-1} is the time-scale of the transformation

$$\tau = \delta_s(\mu)t, \text{ with } \delta_s(\mu) = O(1).$$

Furthermore $\Phi^*(\tau, \mu) = \Phi(\frac{\tau}{\delta_s}, \mu)$. If J^* is the image of J under the transformation $\tau = \delta_s(\mu)t$, then obviously $\|\Phi\|_J = \|\Phi^*\|_{J^*} = \max_{\tau \in J^*} |\Phi^*|$. Furthermore, $\{\Phi(t, \mu) = O(\delta) \text{ in } J\} \iff \{\Phi^*(t, \mu) = O(\delta) \text{ in } J^*\}$. Now we introduce the notion of uniform behaviour.

Definition 3. If for a given time-scale δ_s^{-1} there exists an order function δ such that

$\Phi^*(\tau,\mu) = O(\delta)$ for all bounded closed μ-independent intervals J^* then we shall say that

$\Phi^*(\tau,\mu) = O(\delta)$ uniformly on the time-scale δ_s^{-1}.

Similarly if $\Phi^*(\tau,\mu) = O(\delta)$ for all bounded closed μ-independent intervals J^* then we shall say that

$\Phi^*(\tau,\mu) = O(\delta)$ uniformly on the time-scale δ_s^{-1}.

Definition 4. If there exists an order function σ such that $\Phi^*(\tau,\mu) = O(\delta)$ uniformly on every scale δ_s^{-1}, then we shall say that $\Phi(t,\mu) = O(\delta)$ uniformly for $t \in I_t$. Similarly if $\Phi^*(\tau,\mu) = O(\delta)$ uniformly on every scale, then $\Phi(t,\mu) = O(\delta)$ uniformly in I_t.

Remark. It is easy to show that if, in the sense of Definition 4, a function $\Phi(t,\mu) = O(\delta)$ uniformly in I_t then there exists a constant K such that $|\Phi| \le k\delta, t \in I_t$.

Definition 5. (i) $Z^*(\tau,\mu)$ is a (uniform) asymptotic approximation of $Y^*(\tau,\mu)$ in I_t if $Y^*(\tau,\mu) - Z^*(\tau,\mu) = O(1)$ in I_t where $\tau = \delta_s t$.

(ii) $Z^*(\tau,\mu)$ is a uniform asymptotic approximation of $Y^*(\tau,\mu)$ on a time-scale δ_s^{-1} if $Y^*(\tau,\mu) - Z^*(\tau,\mu) = O(1)$ uniformly on that time scale.

(iii) $Z^*(\tau,\mu)$ is a uniform asymptotic approximation of $Y^*(t,\mu)$ in I_t if $Y^*(\tau,\mu) - Z^*(\tau,\mu) = 0(1)$ uniformly in I_t.

Also we shall need the following definition in the subsequent analysis.

Definition 6. $D_0 \subset D$ is an interior subset if the distance between the boundary of D_0 and the boundary of D is bounded from below by a positive constant, independent of μ for $\mu \in I$.

2. Main Results

Now we have the following result:

Theorem 1. *Assume that*

(i) *the functions Y^1 and Y^2 are define by*

$$\begin{aligned} Y^1(\tau,\mu) &= Y_0^1 + \int_0^\tau F_1(\frac{r}{\mu}, Y^1(r,\mu),\mu)dr \\ Y^2(\tau,\mu) &= Y_0^2 + \int_0^\tau F_2(\frac{r}{\mu}, Y^2(r,\mu),\mu)dr \end{aligned}$$

where $Y_0^1, Y_0^2 \in D_0$, $|Y_0^1 - Y_0^2| \le \delta_0(\mu)$, $\delta_0(\mu) = O(1)$.

(ii) *For all* $Y \in \overline{D}$ *and* $0 \le \tau \le a$

$$|F_1(\frac{\tau}{\mu}) - F_2(\frac{\tau}{\mu}, Y,\mu)| \le \delta_1(\mu), \delta_1(\mu) = O(1).$$

(iii) *The solution* $Y^2(\tau,\mu) \in D_0$ *exists for* $0 \le \tau \le a$. *Then, the solution* $Y^1(\tau,\mu)$ *exists for* $0 \le \tau \le a$, *and in this interval*

$$|Y^1(\tau,\mu) - Y^2(\tau,\mu)| \le \delta_0(\mu)e^{\lambda\tau} + \delta_1(\mu)\frac{1}{\lambda}(e^{\lambda\tau} - 1).$$

where λ *is the Lipschitz constant defined in hypothesis* H_2.

Proof.

$$\begin{aligned}
|Y^1(\tau,\mu) - Y^2(\tau,\mu)| &\le \delta_0(\mu) + \int_0^\tau |F_1(\frac{r}{\mu}, Y^1, \mu) - F_2(\frac{r}{\mu}, Y^2, \mu)|dr \\
&\le \delta_0(\mu) + \int_0^\tau |F_1(\frac{r}{\mu}, Y^1, \mu) - F_1(\frac{r}{\mu}, Y^2, \mu)|dr \\
&\quad + \int_0^\tau |F_1(\frac{r}{\mu}, Y^2, \mu) - F_2(\frac{r}{\mu}, Y^2, \mu)|dr.
\end{aligned}$$

Since Y^1 is a continuous function, it will remain in D_0 for some $0 < \tau \le \tau_1$. Then by using (ii) and the Lipschitz condition we have

$$|Y^1 - Y^2| \le \delta_0(\mu) + \delta_1(\mu) + \lambda \int_0^\tau |Y^1(r,\mu) - Y^2(r,\mu)|dr, \tau \in [0,1].$$

Applying Gronwall's inequality (see Coddengton and Levinson [2]) we get

$$|Y^1 - Y^2| \le \delta_0(\mu)e^{\lambda\tau} + \delta_1(\mu)\frac{1}{\lambda}(e^{\lambda\tau} - 1).$$

The continuity of the solution Y^2 is obtained from lemma B and Y^2 will remain in D_0 for μ sufficiently small. For every continuation the above estimate of $|Y^1 - Y^2|$ remains valid as long as Y^2 remains in D_0. Therefore the continuation and the estimate are valid in $0 \le \tau \le a$. This completes the proof.

In this paper the essential tool for the study of such problems is the concept of local average value that will be introduce now.

Definition 7. Consider the function $\Phi(t,\mu)$ and a transformation $\tau = \delta_s(\mu)t$, $\Phi(\frac{\tau}{\delta_s},\mu) = \phi^*(\tau,\mu)$. A local average value $\widetilde{\Phi}(\tau,\mu)$ of $\Phi^*(\tau,\mu)$ on the time-scale δ_s^{-1} is given by

$$\widetilde{\Phi}(\tau,\mu) = \frac{1}{\delta(\mu)} \int_0^{\delta(\mu)} \Phi^*(\tau + \tau', \mu)d\tau' \tag{2.1}$$

where δ is some order function with $\delta(\mu) = O(1)$.

Remark. In the above definition, the function $\Phi(t, \mu)$ is in fact averaged (in the usual sense of the word) over a (small) distance on the δ_s^{-1} time scale. The "smallness" of the distance over which the average is performed is in asymptotic sense, and is measured by the order function $\delta(\mu)$. It is obvious that a "small" distance on the δ_s^{-1} time-scale may be a "large" distance in the original t time variable. The average $\tilde{\Phi}(\tau, \mu)$ depends on the choice of $\delta(\mu)$, which leaves us with a degree of liberty to be exploited later, in the analysis. Naturally, $\tilde{\Phi}(\tau, \mu)$ also depends on the time-scale δ_s^{-1}, on which $\Phi(t, \mu)$ is investigated. The asymetry in the definition of $\phi(\tau, \mu)$ ("forward" integration, $\tau' \geq 0$) is chosen, because otherwise $\tilde{\Phi}(0, \mu)$ could not be defined. Finally we remark that for the purpose of calculation, it is often advantageous to introduce an obvious change of integration variable $\tau' = \delta\tau$ which yields.

$$\tilde{\Phi}(\tau, \mu) = \int_0^1 \Phi^*(\tau + \delta\tilde{\tau}, \mu) d\tilde{\tau}.$$

The usefulness of the local average values immediately appears from the following fundamental result on the natural time scale $\tau = \mu t$.

Lemma 1. *Let $Y^*(\tau, \mu)$ be the solution in the interval $0 \leq \tau \leq a$ of the system (1.2), then*

$$\text{(i)} \quad |Y^*(\tau + \delta\tilde{\tau}, \mu) - Y^*(\tau, \mu)| \leq M\delta\tilde{\tau} \tag{2.2}$$

$$\text{(ii)} \quad Y^*(\tau, \mu) = \tilde{Y}(\tau, \mu) + O(\delta) \tag{2.3}$$

where $M = \sup_G |F|$.

Proof. (i) From the system (1.2) it follows that

$$\begin{aligned} |Y^*(\tau + \delta\tilde{\tau}, \mu) - Y^*(\tau, \mu)| &\leq \int_\tau^{\tau+\delta\tilde{\tau}} |F(\frac{r}{\mu}, Y^*(r, \mu), \mu)| dr \\ &\leq M\delta\tilde{\tau} \end{aligned}$$

(ii) Using the definition of average values (2.1) we get

$$\begin{aligned} \tilde{Y}(\tau, \mu) &= \int_0^1 Y^*(\tau + \delta\tilde{\tau}, \mu) d\tilde{\tau} \\ &= Y^*(\tau, \mu) + \int_0^1 |Y^*(\tau + \delta\tilde{\tau}, \mu) - Y^*(\tau, \mu)| d\tilde{\tau}. \end{aligned}$$

Using (2.2) we get

$$|Y(\tau,\mu)-Y^*(\tau,\mu)|\le\frac{1}{2}M\delta$$

which completes the proof.

Now we investigate local averages of the functions $Y^*(\tau,\mu)$ defined as solutions of the system (1.2). It is evident that the system (1.2) is equivalent to the integral system

$$Y^*(\tau,\mu)=Y_0+\int_0^{\tau}F(\frac{r}{\mu},Y^*(r,\mu),\mu)dr. \tag{2.4}$$

According to (2.1) we find, for the local average

$$\widetilde{Y}(\tau,\mu)=Y_0+\int_0^1\{\int_0^{\tau+\delta\tilde{\tau}}F(\frac{r}{\mu},Y^*(r,\mu),\mu)dr\}d\tilde{\tau}. \tag{2.5}$$

We shall deduce a relation for $\widetilde{Y}(\tau,\mu)$ does not contain $Y^*(\tau,\mu)$, in the following result.

Theorem 2. *If*

$$Y^*(\tau,\mu)=Y_0+\int_0^{\tau}F(\frac{r}{\mu},Y^*(r,\mu),\mu)dr$$

then

$$\widetilde{Y}(\tau,\mu)=Y_0+\int_0^{\tau}\int_0^1F(\frac{\tau'}{\mu}+\frac{\delta}{\mu}\tilde{\tau},\widetilde{Y}(r,\mu),\mu)d\tilde{\tau}dr+I_1+I_2$$

where

$$|I_1|\le\frac{1}{2}M\delta,\quad |I_2|\le\lambda M\delta\tau.$$

Proof. We rewrite the right hand side of (2.5) as follows

$$\widetilde{Y}(\tau,\mu)=Y_0+\int_0^1[\int_{\delta\tilde{\tau}}^{\tau+\delta\tilde{\tau}}F(\frac{r}{\mu},Y^*(r,\mu),\mu)dr]d\tilde{\tau}+I_1 \tag{2.6}$$

where

$$I_1=\int_0^1\int_0^{\delta\tilde{\tau}}F(\frac{r}{\mu},Y^*(r,\mu),\mu)drd\tilde{\tau}.$$

It is immediately obvious that

$$|I_1| \leq \frac{1}{2} M\delta.$$

For the first integral on the right side of (2.6), we make the substitution $r = \tau' + \delta\tilde{\tau}$ and subsequently we interchange the order of integration. It follows that

$$\widetilde{Y}(\tau,\mu) = Y_0 + \int_0^\tau [\int_0^1 F(\frac{\tau'}{\mu} + \frac{\delta}{\mu}\tilde{\tau}, Y^*(\tau' + \delta\tilde{\tau},\mu),\mu)d\tilde{\tau}]d\tau' + I_1.$$

Finally we can write

$$\widetilde{Y}(\tau,\mu) = Y_0 + \int_0^\tau [\int_0^1 F(\frac{\tau'}{\mu} + \frac{\delta}{\mu}\tilde{\tau}, \widetilde{Y}(\tau',\mu),\mu)d\tilde{\tau}]d\tau' + I_1 + I_2$$

where

$$I_2 = \int_0^\tau [\int_0^1 F(\frac{\tau'}{\mu} + \frac{\delta}{\mu}\tilde{\tau}, Y^*(\tau' + \delta\tilde{\tau},\mu),\mu) - F(\frac{\tau'}{\mu} + \frac{\delta}{\mu}\tilde{\tau}, \widetilde{Y}(\tau',\mu),\mu)d\tilde{\tau}]d\tau'.$$

In order to estimate I_2 we use Lipschitz-continuity of F

$$|I_2| \leq \lambda \int_0^\tau \int_0^1 |Y^*(\tau',\delta\tilde{\tau},\mu) - \widetilde{Y}(\tau',\mu)|d\tilde{\tau}d\tau'$$

and using lemma 1 we get

$$|I_2| \leq \lambda M\delta\tau$$

and the result follows.

Using Theorem 1 and Lemma 1 (ii) we have the following result:

Theorem 3. *If*

$$Y^*(\tau,\mu) = Y_0 + \int_0^\tau F(\frac{r}{\mu}, Y^*(r,\mu),\mu)dt$$

then the local average $\widetilde{Y}(\tau,\mu)$ of $Y^*(\tau,\mu)$ can be approximated by the function $Y^{\approx}(\tau,u)$ satisfying:

$$Y^{\approx}(\tau,\mu) = Y_0 + \int_0^\tau \int_0^1 F(\frac{\tau'}{\mu} + \frac{\delta}{\mu}\tilde{\tau}, Y^{\approx}(\tau',\mu),\mu)d\tilde{\tau}d\tau'.$$

We have

$$\tilde{Y}(\tau,\mu) = Y^{\approx}(\tau,\mu) + O(\delta),$$

and

$$Y^{*}(\tau,\mu) = Y(\tau,\mu) + O(\delta),$$

the estimates being valid on every closed interval on which $\widetilde{Y}(\tau,\mu)$ exists and $Y^{\approx} \in D_0$.

Remark. This theorem is the general and fundamental result, permitting approximation of $Y^{*}(\tau,\mu)$ by a function which is an approximation of the local average of $Y^{*}(\tau,\mu)$.

Definition 8. The function $F(t,Y,\mu)$ is said to be a Krilov-Bogolioubov-$(K-B)$-function if

$$F_0(Y,\mu) = \lim_{T\to\infty}(1/T)\int_0^T F(t,Y,\mu)dt$$

exists.

In the sense of definition 7, a local average value of $F(\tau/\delta_s, Y^{*},\mu)$ is given by

$$F(\tau/\delta_s, Y^{*},\mu) = (1/\delta(\mu))\int_0^{\delta(\mu)} F((\tau/\delta_s)+(r/\delta_s), Y^{*},\mu)dr,$$

where $\delta_s(\mu) = O(1)$.

Theorem 4. *Let the system*

$$dY^{*}/d\tau = F(\tau/\mu, Y^{*},\mu)$$

have the solution $Y^{*}(\tau,\mu)$, with the condition $Y^{*}(0,\mu) = Y_0 \in D_0$, and the system

$$d\eta/d\tau = F_0(\eta),$$

have the solutio $\eta(\tau)$, with $\eta(0) = Y_0 \in D_0$, where,

$$F_0(\eta) = \lim_{T\to\infty} 1/T\int_0^T F(t,\eta,0)dt, \quad \eta \in D_0.$$

Suppose that $\eta(\tau)$ exists for $\tau \in [0,a]$ and $\mu \in I_\mu$, then $Y^{*}(\tau,\mu)$ exists in the same interval and

$$Y^{*}(\tau,\mu) = \eta(\tau) + O(1).$$

where F is assumed to be $(K - B)$ function.

Proof. By hypothesis H_1 there exists an order function $\delta_2(u) = O(1)$ such that for all $Y \in D$ and $t \in I_t$

$$|F(t, Y, \mu) - F(t, Y, 0)| \leq \delta_2(\mu), \delta_2(u) = O(1).$$

It is clear that

$$Y^{\approx}(\tau, \mu) = Y_0 + \int_0^{\tau} F_0(Y^{\approx}(r))dr + I_3,$$

where

$$|I_3| = M_1\{(\delta_1/\delta) + \delta_2\}\tau, \delta_1/\delta = O(1)$$

and M_1 is a constant independent of μ. Applying Theorem 1, we find that

$$|Y^{\approx}(\tau, \mu) - \eta(\tau)| \leq (1/\lambda)M_1(\delta_1/\delta + \delta_2)(e^{\lambda\tau} - 1).$$

Hence, for any interval $0 \leq \tau \leq a$, for which $\eta(\tau)$ exists, $Y^{\approx}(\tau, \mu)$ exists and

$$Y^{\approx}(\tau, \mu) = \eta(\tau) + O(\delta_1/\delta) + O(\delta_2).$$

It follows, from Theorem 3, that

$$Y^*(\tau, \mu) = \eta(\tau) + O(\delta_1/\delta) + O(\delta) + O(\delta_2).$$

Since $\delta(\mu) = O(1)$ is an order function such that $\delta_1/\delta = O(1)$; $\eta(\tau)$, indeed, is an asymptotic approximation of $Y^*(\tau, \mu)$, that is

$$Y^*(\tau, \mu) = \eta(\tau) + 0(1).$$

This completes the proof.

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MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, AL-AZHAR UNIVERSITY, NASR-CITY, CAIRO-EGYPT.