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On the Induced Complex Characters of Finite Groups

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Representation theory and character theery provide major tools for the study of finite groups. Complex representations and their characters were first studied nearly one hundred years ago by Frobenius.

The purpose of this paper is to prove a property of induced characters with respect to finite groups [Theorem 3]. Throughout this paper, we shall assume that

(1) every group G means a finite group without any statements,

(2) $N \lhd G$ means that N is a normal subgroup of G,

(3) every character of a group G means a C - *character* unless otherwise stated, where C is the set of all complex numbers,

(4) for a group G, $I_{rr}(G)$ is the set of all irreducible characters of G. Moreover, for a subgroup H of a group G, we shall use the following notations,

(5) for a character χ of G, χ_H is the restriction of χ to H,

(6) for a character θ of H, θ^G is the induced character ([1]~[5]) of G.

Lemma 1. *Let $N \lhd G$ and $\chi \in I_{rr}(G)$. Then $(\chi_N)^G = \rho\chi$ where ρ is a regular character of G which is defined by $\rho(x) = \rho'(N_x)(x \in G)$ for a regular character ρ' of G/N and $\rho\chi(x) = \rho(x)$ $\chi(x)$.*

Proof. Since χ is a class function, for $x \in N$ it follows that

$$\begin{aligned}(\chi_N)^G(x) &= \frac{1}{|N|}\sum_{g\in G} \chi_N^0(gxg^{-1}) \\ &= \frac{1}{|N|}\sum_{g\in G} \chi_N(x^g) \\ &= \frac{1}{|N|}\sum_{g\in G} \chi_N(x) \\ &= (|G|/|N|)\chi_N(x)\ ,\end{aligned}$$

where

$$\chi_N^G(h) = \begin{cases} \chi_N(h) & \text{if } h \in N \\ 0 & \text{if } h \notin N \end{cases}$$

and $x^g = gxg^{-1}$. Thus we have

$$(\chi_N)^G(x) = \begin{cases} (|G|/|N|)\chi(x) & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

On the other hand, let ρ' be a regular character of G/N. Then

$$\rho'(N_x) = \begin{cases} |G|/|N| & \text{if } N_x = N \\ 0 & \text{if } N_x \neq N \end{cases} \quad ([1], [4], [5]).$$

Hence

$$\rho(x) = \rho'(N_x) = \begin{cases} |G|/|N| & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

is a regular character of G ([5]). Therefore $(\chi_N)^G = \rho\chi$.

Definition 2. *For a group G let $\chi \in I_{rr}(G)$. If, for any character θ of a proper subgroup of G, $\chi \neq \theta^G$, then χ is called a primitive character of G.*

In the above situation, if $(\chi_N, \theta) > 0$, then θ is called *an irreducible constituent of χ_N*, where we have to note that

(1) $N \triangleleft G$, $\chi \in I_{rr}(G)$ and $\theta \in I_{rr}(N)$

(2) $(\chi_N, \theta) = (\chi_N, \theta)_N = \frac{1}{|N|}\sum_{x \in N} \chi_N(x)\theta(x^{-1})$ and $|N|$ = the order of N.

Theorem 3. *Let $N \triangleleft G$ and G/N be an abelian group. If, for a primitive character χ of G, $\theta \in I_{rr}(N)$ is a constituent of χ_N, then either*

(1) $\theta^G = e\chi$ where $e^2 = |G:N|$ or

(2) $\theta^G = \sum_{i=1}^{s} \chi_i$, where $\chi_1 = \chi, \chi_2, \cdots, \chi_s$ are the distinct irreducible characters and $s = |G:N|$.

Proof. Let θ be an irreducible constituent of χ_N such that $(\chi_N, \theta) = e > 0$. Then since χ is a primitive character of G, we have

$$(*) \qquad \chi_N = e\theta \qquad ([5]).$$

On the other hand, since G/N is an abelian group, we may put $\{\lambda_1', \lambda_2', \cdots, \lambda_s'\} = I_{rr}(G/N)$ and $s = |G:N|$, where $\lambda_1', \lambda_2', \cdots, \lambda_s'$ are the linear characters of G/N, and for $i = 1, 2, \cdots, s$, we put $\chi_i = \lambda_i\chi$, where λ_i are the linear character of G such that $\lambda_i(x) = \lambda_i'(N_x)$. Then $\chi_1, \chi_2, \cdots, \chi_s$ are the irreducible characters of G and, since for $i = 1, 2, \cdots, s$, $ker\,\lambda_i \supset N$, $\chi_{iN} = \chi_N$. Thus by Frobenious reciprocity ([1]~[5]), we have

$$(\ast\ast) \qquad (\chi_i, \theta^G) = e$$

for $i = 1, 2, \cdots, s$.

Hence $\chi_1, \chi_2, \cdots, \chi_s$ are the irreducible constituents of θ^G. Furthermore, since $\rho = \sum_{i\in T}^{s}\lambda_i$, from Lemma 1 we have

$$\begin{aligned}(\chi_N)^G = \rho\chi &= (\sum_{i\in T}^{s}\lambda_i)\chi \\ &= \sum_{i\in T}^{s}\lambda_i\chi \\ &= \sum_{i\in T}\chi_i.\end{aligned}$$

For any $\xi \in I_{rr}(G)$ with $\xi \notin \{\chi_1, \chi_2, \cdots, \chi_s\}$ we have

$$\begin{aligned}(\xi, (\chi_N)^G) &= (\xi, \sum_{i\in T}^{s}\chi_i) \\ &= \sum_{i\in T}^{s}(\xi, \chi_i) \\ &= 0.\end{aligned}$$

Now, $\chi_N = e\sum_{i\in T}^{t}\theta_i$, where $\theta = \theta_1, \theta_2, \cdots, \theta_t$ are the distinct conjugates in G ([1], [2], [4], [5]).

Hence, by Frobenious reciprocity it follows that

$$\begin{aligned}0 = (\xi, (\chi_N)^G) &= (\xi_N, \chi_N) \\ &= (\xi_N, e\sum_{i\in T}^{t}\theta_i) \\ &= e\sum_{i\in T}^{t}(\xi_N, \theta_i) \\ &= e\sum_{i\in T}^{t}(\xi, \theta_i^G),\end{aligned}$$

and so since $(\xi, \theta_i^G) \geqq 0$ for $i = 1, 2, \cdots, t$, we obtain $(\xi, \theta^G) = 0$. This implies that all the irreducible constituents of θ^G are in the set $\{\chi_1, \chi_2, \cdots, \chi_s\}$.

Therefore, by $(\ast\ast)$ we have

$$(\ast\ast\ast) \qquad \theta^G = e\sum_{i\in T}^{s}\chi_i$$

where $s = |G:N|$.

First, suppose that $\chi_1, \chi_2, \cdots, \chi_s$ are distinct. Then, by $(\ast\ast\ast)$ and the definition of induced character of G, we have $|G:N|\theta(1) = \theta^G(1) = e|G:N|\chi(1)$, and, by $(\ast)$, $\chi(1) = \chi_N(1) = e\theta(1)$. Thus we obtain $e = 1$. Therefore

$$\theta^G = \sum_{i\in T}^{s}\chi_i .$$

This is the situation (2).

In the remainder case, since $\chi_1, \chi_2, \cdots, \chi_s$ are not distinct, we may put $\chi_i = \lambda_i\chi = \lambda_j\chi = \chi_j$ for some λ_i, λ_j with $\lambda_i \neq \lambda_j$ $(i, j = 1, 2, \cdots, s)$. Then from $(\ast\ast)$ and $(\ast\ast\ast)$ we have

$$e = (\chi_i, \theta^G) = (\chi_i, e\sum_{i\in T}^{s}\chi_i) = 2e$$

for $i = 1, 2, \cdots, s$. Since e is a positive integer, this is impossible. So χ is a unique irreducible constituent of θ^G, and thus $\theta^G = e\chi$. Moreover, we have $|G:N|\theta(1) = \theta^G(1) = e\chi(1)$ and $\chi_N(1) = e\theta(1)$. Thus we obtain $|G:N| = e^2$. This is the situation (1) and the proof is complete.

Corollary 4. *Under the hypotheses of Theorem 3 let $|G:N| = p$ be a prime number. Then*

$$\theta^G = \sum_{i\in T}^{s}\chi_i ,$$

where $\chi_1, \chi_2, \cdots, \chi_s$ are the distinct irreducible characters and $s = |G:N|$.

Proof. In Theorem 3, the case (1) can't be occurred since p is not a square.

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