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Iterative Algorithms for General Quasi Complementarity Problems

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Abstract

In this paper, we consider an iterative algorithm for solving a new class of quasi complementarity problems of finding $u \in R^n$ such that

$$g(u) \in K(u),\ Tu + A(u) \in K^*(u),\ \text{ and } \langle g(u),\ Tu + A(u) \rangle = 0,$$

where T, A and g are continuous mappings from R^n into itself and $K^*(u)$ is the polar cone of the convex cone $K(u)$ in R^n. The algorithms considered in this paper are general and unifying ones, which include many existing algorithms as special cases for solving the complementarity problems. We also study the convergence criteria of the general algorithms.

1. Introduction

Variational inequality theory provides us not only a general and unified framework to study a wide class of nonlinear problems arising in various branches of mathematical and engineering sciences, but also gives us new numerical methods for solving them. Closely related to the variational inequality problem is the complementarity problem, which plays an important and fundamental role in general equilibrium theory of economics and transportation, management sciences, and operations research. In recent years, various useful extensions of these two different problems have been introduced and analyzed, see Bensoussan and Lions [4], Baiocchi and Capelo [3], Karamardian [11], Pang [28], Noor [19, 25], and the references therein.

In this paper, we introduce and study a new class of quasi complementarity problems, which is called the general mildly (strongly) nonlinear quasi complementarity problem. Using the variational inequality technique, we propose and analyze a new and unified iterative algorithm. We also study the conditions under which the approximate solution obtained from the iterative algorithm converges to the exact solution. Several special cases, which can be obtained from our main results are also discussed.

In section 2, we introduce new classes of variational inequality and complementarity problems and discuss several special cases. Algorithms and convergence results are considered and discussed in section 3 and 4.

2. Formulation and Basic Results

We denote the inner product and norm on R^n by $\langle .,. \rangle$ and $\| . \|$, respectively.
Let K be a closed convex cone in R^n and T, $g : R^n \to R^n$ be the continuous mappings. Given a nonlinear mapping $A : R^n \to R^n$ and a point-to-set mapping $K : u \to K(u)$, which associates a closed convex set $K(u)$ of R^n with any element u of R^n, consider a problem of finding $u \in R^n$ such that $g(u) \in K(u)$ and

$$\langle Tu+A(u),\ g(v)-g(u)\rangle \geqq 0, \quad \text{for all } g(v) \in K(u) \tag{2.1}$$

problem of the type (2.1) is known as the general mildly (strongly) nonlinear quasi variational inequality problem, which is mainly due to Noor [13].

Note that if $K(u) \equiv K$, then problem (2.1) is equivalent to finding $U \in R^n$ such that $g(u) \in K$ and

$$\langle Tu + A(u), g(v) - g(u)\rangle \geqq 0, \text{ for all } g(v) \in K, \tag{2.2}$$

which is called the general strongly nonlinear variational inequality problem. For applications and iterative methods see Noor [12, 14].

Example 2.1.

It is worth mentioning that a large number of unrelated general equilibrium problems, moving and free boundary value problems of odd order can be studied in the general framework of the variational inequality problems (2.1) and (2.2). As an example of odd order free boundary value problem leading to general quasi variational inequality problem (2.1), we consider the third order obstacle problem

$$\begin{cases} Tu \geqq f(x, u(x)), & \text{in } \Omega \\ u(x) \geqq Mu(x), & \text{in } \Omega \\ [Tu - f(x, u(x))]\,[u(x) - Mu(x)] = 0 & \text{in } \Omega \\ u(0) = 0,\ u'(0) = 0 = u'(1) & \end{cases} \tag{2.3}$$

where $\Omega = [0, 1]$ is a domain, $T = (-d^3/dx^3)$ is the differential operator of third order, f is a given nonlinear function of x and u(x), and Mu(x) is the obstacle function, where M is an operator of the form

$$Mu = 1 + \inf_{\substack{\xi \geqq 0 \\ x+\xi \in \Omega}} u(x+\xi),\quad x \in \Omega,$$

see Mosco [10] for further details. To study the problem (2.3) via the variational inequality technique, we define $K(u) = \{u \in H_0^2(\Omega),\ u(x) \geqq Mu(x) \text{ on } \Omega\}$, which is a closed convex set in $H_0^2(\Omega)$, see Oden and Kikuchi [26], for the definition of the space $H_0^m(\Omega)$. Now using the technique of K-positive definite operators, as developed in [30], we can show that the problem (2.3) is equivalent to finding $u \in H_0^2(\Omega)$ such that $g(u) \in K(u)$ and

$$\langle Tu + A(u), g(v) - g(u) \rangle \geqq 0, \quad \text{for all } g(v) \in K(u),$$

where

$$\langle Tu, g(v) \rangle = -\int_0^1 (D^3u)\,(Dv)\,dx = \int_0^1 (D^2u)\,(d^2v)\,dx,$$

and

$$\langle A(u), g(v) \rangle = \int_0^1 f(x, u(x))\,Dv\,dx,$$

with

$$g = \frac{d}{dx} = D.$$

It is clear that with $g = (d/dx)$, we have the variational inequality problem (2.1). Well-known examples of obstacle, unilateral, moving and free boundary value problems, which may be written in the form (2.1) or (2.2) includes fluid flow through porous media, journal bearing lubrication problems, contact problems in elasticity, etc., where $f(u) = f(x, u(x))$ is of the form e^{-u}, $e^{u}-1$, u^{n}, $n \geqq 2$.

Related to the general mildly (strongly) nonlinear quasivariational inequality problems, we now introduced a new class of complementarity problems, which will be called general mildly (strongly) nonlinear quasi complementarity problem of finding $u \in R^n$ such that

$$g(u) \in K(u), \quad Tu + A(u) \in K^*(u), \quad \langle g(u), Tu + A(u) \rangle = 0 \tag{2.4}$$

where $K^*(u)$ is the polar cone of the convex cone $K(u)$ in R^n. In many important applications, $K(u)$ has the form

$$K(u) = m(u) + K, \tag{2.5}$$

where m is a point-to-point mapping.

Remark 2.1

For different choices of the mapping T, g, A and the convex sets K, we may obtain various previously known results considered by many authors including Baiocchi and Capelo [3], Bensoussan and Lions [4], Chan and Pang [5], Crank [6], Glowinski, Lions and Tremolieres [7], Lions and Stampacchia [9], Noor [11, 15, 16, 17, 18, 19, 20, 21, 22], Oden and Kikuchi [26] and Pang [28, 29].

3. Iterative Algorithms

We need the following results in order to suggest an algorithm for finding the approximate solution of the general mildly (strongly) nonlinear quasi complementarity problem (2.4). The first is a generalization of a result of Noor [15].

Lemma 3.1. If $K(u)$ is a positive cone in R_n and $K \subset g(K)$, then $u \in K(u)$, defined by (2.5), is a solution of general mildly (strongly) nonlinear quasi complementarity problem (2.4), if and only if u satisfies the general mildly (strongly) nonlinear quasi variational inequality problem (2.1).

Lemma 3.2. For $K(u)$ given by (2.5), $u \in R^n$ is a solution of the general mildly (strongly) nonlinear quasi variational inequality problem (2.1), if and only if u satisfies the relation

$$u = F(u),$$

where

$$F(u) = u-g(u) + m(u) + P_K[g(u)-\rho(Tu+A(u)-m(u)] \tag{3.1}$$

for some $\rho > 0$, where m is an arbitrary point-to-point mapping and P_K is the projection of R^n into K.

Proof. Suppose that $u \in R^n$ satisfies (2.1), then

$$\langle Tu+A(u), g(v)-g(u)\rangle \geqq 0, \qquad \text{for all } g(v) \in K(u).$$

For some $\rho > 0$, we can rewrite the above inequality into the following form

$$(\rho\langle Tu + A(u), g(v)-g(u)\rangle) \geqq 0,$$

which is equivalent to finding $u \in R^n$ such that

$$\langle g(u)-(g(u)-\rho(Tu+A(u))), g(v)-g(u)\rangle \geqq 0, \qquad \text{for all } g(v) \in K(u).$$

Hence by Lemma 3.1 [11], we have

$$g(u) = P_{K(u)}[g(u)-\rho(Tu+A(u))].$$

Using the fact that $P_{K(u)}(v) = m(u) + P_K[v-m(u)]$, we get

$$g(u) = m(u) + P_K[g(u) -\rho(Tu + A(u)) -m(u)]$$

or equivalently

$$u = u - g(u) + m(u) + P_K[g(u) - \rho(Tu + A(u)) - m(u)].$$

which is the required result.

From lemma 3.1 and lemma 3.2, we conclude that the solution of problem (2.4) may be obtained by computing the fixed point of the function defined by (3.1). This formulation is very useful in approximation and numerical analysis of the complementarity problems. One of the consequences of this formulation is that we can obtain an approximate solution by an iterative algorithm.

Remark 3.1. We would like to emphasize that the fixed point formulation (3.1) of the problem (2.4) is symmetric with respect to the mappings T and g. To show this fact, we need the well-known [32] result that for all $z \in R^n$

$$\begin{aligned} z &= P_K(z) + P_{-K^*}(z) \\ &= P_K(z) - P_{K^*}(-z). \end{aligned}$$

We may rewrite (3.1) as

$$F(u) = S(u) + P_K[G(u) - S(u)] \tag{3.2}$$

with $S(u) = u - g(u) + m(u)$,
and

$$G(u) = u - \rho(Tu + A(u)).$$

Taking $z = G(u) - S(u)$, we obtain

$$0 = G(u) - S(u) - P_K[G(u) - S(u)] + P_{K^*}[S(u) - G(u)] \tag{3.3}$$

Adding (3.2) and (3.3), we obtain

$$F(u) = G(u) + P_{K^*}[S(u) - G(u)], \tag{3.4}$$

which shows that F is implicitly symmetric.

On the basis of these observations, we now suggest and analyze a general and unified algorithm for the problem (2.4) as:

ALGORITHM 3.1

Given $u_0 \in R^n$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} u_{n+1} &= F(u_n) \\ &= u_n - g(u_n) + m(u_n) + P_K[g(u_n) - \rho(Tu_n + A(u_n)) - m(u_n), \quad n = 0, 1, 2, \cdots, \end{aligned}$$

where $\rho > 0$ is a constant.

If T is a linear affine mapping of the type $T: u \rightarrow Mu+q$, for $M \in R^{n\times n}$ and $q \in R^n$, then algorithm 3.1 can be written in the form:

ALGORITHM 3.2

For any given $u_0 \in R^n$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K[g(u_n) - \rho E\{Mu_n + q + L(u_{n+1} - u_n) + A(u_n)\} - m(u_n)], \quad \text{for } n = 0, 1, 2, \cdots\cdots, \tag{3.6}$$

where $\rho > 0$ is a constant, E is a positive diagonal matrix, and L is either a strictly lower or strictly upper triangular matrix. This restriction on L may be relaxed, because the iterate u_{n+1} may be obtained by solving a variational inequality subproblem as pointed out in Pang [29]. Here the original data M remain intact throughout iteration, allowing this algorithm to be efficient for both large scale and specially structured problems. It is also clear that each iteration of algorithm 3.1 and algorithm 3.2 is itself equivalent to a general mildly (strongly) nonlinear quasi variational inequality problem as implied by lemmas 3.1 and 3.2.

Remark 3.2. The algorithms 3.1 and 3.2 proposed in this paper are general than and include several previously known algorithms as special cases, see [1, 2, 5, 8, 15, 19, 22, 23, 24, 25] for more details.

4. Convergence Analysis

In this section, we consider the convergence properties of algorithms 3.2 and 3.1.

Here we only consider the special case, when $K = [0, b]$ is a closed convex set in R^n. In this case, we consider the projection operator P_k, which is defined as

$$P_k(u) = \arg\min_{v \in K} \| v-u \|.$$

If $K = R^n_+$, then

$$(P_k(u))_i = \max\{0, u_i\}, \quad i = 1, 2, \cdots, n.$$

In our case, we have

$$\begin{aligned}(P_k(u))_i &= (P_{[0,b]}(u))_i \\ &= \min\{\max(0, u_i), b_i\}, \quad i = 1, 2, \cdots, n, \text{ where } b=(b_1, b_2, \cdots,).\end{aligned}$$

For notational purpose, $P_{[0,b]}$ will be denoted as P_k. The operator P_k has the following properties.

Lemma 4.1. [1] For any u and v in R^n,

(i) $u \leq v$ implies $P_k(u) \leq P_k(v)$

(ii) $P_k(u) - P_k(v) \leq P_k(u-v)$

(iii) $P_k(u+v) \leq P_k(u) + P_k(v)$

(iv) $P_k(u) + P_k(-u) \leq |u|$; with equality if and only if $-b \leq u \leq b$.

In addition, the following concepts are also needed. A real matrix $M \in R^{n \times n}$ is said to be Z-matrix (a P-matrix), if it has nonpositive off-diagonal entries (positive principal minors). A square matrix with non-positive off-diagonal elements and with a non-negative inverse is called an M-Matrix. It can be shown that a matrix which is both a Z-matrix and P-matrix is an M-matrix [29], see [31] for full details. If $M \in R^{n \times n}$, then $|M|$ denotes the matrix obtained from M by replacing each element M_{ij} by its absolute value.

Definition 4.1. Let $f: K \subset R^n \to R^n$. We say f is:

(i) strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle f(u) - f(v), u-v \rangle \geq \alpha \| u-v \|^2, \quad \text{for all } u, v \in K$$

(ii) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\| f(u) - f(v) \| \leq \beta \| u-v \|, \text{ for all } u, v \in K.$$

In particular, it follows that $\alpha \leq \beta$.

Theorem 4.1. Suppose that there exist nonnegative matrices $W \in R^{n \times n}$, $N \in R^{n \times n}$, and $B \in R^{n \times n}$ such that

$$|m(u) - m(v)| \leq W|u-v|, \quad \text{for all } u, v, \tag{4.1}$$

$$|A(u) - A(v)| \leq N|u-v|, \quad \text{for all } u, v, \tag{4.2}$$

$$|g(u) - g(v)| \leq B|u-v|, \quad \text{for all } u, v. \tag{4.3}$$

If $\{u_{n+1}\}$ and $\{U_n\}$ are the sequences generated by algorithm 3.2, then

$$|u_{n+1}-u_n| \leq (I-\rho E|L|)^{-1} [2(I+B+W) + \rho EN + |I-\rho E(M-L)|] |u_n-u_{n-1}|. \tag{4.4}$$

and

$$|u_{n+1}-u| \leq (I-\rho E|L|)^{-1} [2(I+B+W) + \rho EN + |I-\rho E(M-L)|] |u_n-u|. \tag{4.5}$$

for each n, where u is the solution of the problem (2.4) and L is either strictly lower or upper triangular matrix.

Proof. From algorithm 3.2 and lemma 4.1, we have

$$\begin{aligned} u_{n+1}-u_n \leq\ & u_n-u_{n-1}-(g(u_n)-g(u_{n-1})) + m(u_n)-m(u_{n-1}) \\ & + P_K[g(u_n)-g(u_{n-1}) + \{I-\rho E(M-L)\} (u_n-u_{n-1}) \\ & -\rho EL(u_{n+1}-u_n)-(u_n-u_{n-1})-(m(u_n)-m(u_{n-1})) \\ & + \rho E(A(u_n)-A(u_{n-1}))]. \end{aligned}$$

Again invoking lemma 4.1 and using the fact that $P_K^2 = P_K$, we obtain

$$P_K[u_{n+1}-u_n-\{(u_n-u_{n-1})-(g(u_n)-g(u_{n-1}))+(m(u_n)-m(u_{n-1}))\}]$$
$$\leq P_K[\{I-\rho E(M-L)\}(u_n-u_{n-1})-\rho EL(u_{n+1}-u_n)+\rho E(A(u_n)-A(u_{n-1}))$$
$$-(u_n-u_{n-1})-(m(u_n)-m(u_{n-1}))+(g(u_n)-g(u_{n-1}))] \tag{4.6}$$

In a similar way, we have

$$P_K[-((u_{n+1}-u_n)-\{(u_n-u_{n-1})-(g(u_n)-g(u_{n-1}))+(m(u_n)-m(u_{n-1}))\}]$$
$$\leq P_K[\{-I+\rho E(M-L)\}(u_n-u_{n-1})+\rho EL(u_{n+1}-u_n)-\rho E(A(u_n)-A(u_{n-1}))$$
$$+(u_n-u_{n-1})-(m(u_n)-m(u_{n-1}))+(g(u_n)-g(u_{n-1}))] \tag{4.7}$$

Adding the inequalities (4.6) and (4.7), and using lemma 4.1, we have

$$|u_{n+1}-u_n| \leq 2|u_n-u_{n-1}|+2|g(u_n)-g(u_{n-1})|+2|m(u_n)-m(u_{n-1})|$$
$$+|I-\rho E(M-L)|\,|u_n-u_{n-1}|+\rho E|L|\,|u_{n+1}-u_n|+\rho E|A(u_n)-A(u_{n-1})|$$

using (4.1), (4.2), and (4.3), we have

$$|u_{n+1}-u_n| \leq [2(I+B+W)+\rho EN+|I-\rho E(M-L)|]\,|u_n-u_{n-1}|+\rho E|L|\,|u_{n+1}-u_n|$$
$$(I-\rho E|L|)\,|u_{n+1}-u_n| \leq [2(I+B+W)+\rho EN+|I-\rho E(M-L)|]\,|u_n-u_{n-1}|.$$

Since L is either a strictly lower or upper triangular matrix, so the matrix $(I-\rho E|L|)$ is invertible and its inverse is nonnegative, that is, $(I-\rho E|L|)$ is an M-matrix. Hence

$$|u_{n+1}-u_n| \leq (I-\rho E|L|)^{-1}[2(I+B+W)+\rho EN+|I-\rho E(M-L)|]\,|u_n-u_{n-1}|,$$

which is the required result (4.4). Using similar arguments, we can obtain (4.5).

From theorem 4.1, we can obtain a sufficient condition for the convergence of the sequence $\{u_{n+1}\}$ generated by algorithm 3.2 to be bounded and hence have an accumulation point, which is the solution of the general mildly (strongly) nonlinear quasi complementarity problem (2.4).

Theorem 4.2. Assume that

$$\sigma(G) < 1\ ,$$

where

$$G = (I-\rho E|L|)^{-1}\,[2(I+B+W) + \rho EN + |I-\rho E(M-L)|], \tag{4.8}$$

with σ denoting the spectral radius. Then for any initial vector u_0, the sequence $\{u_{n+1}\}$ generated by algorithm 3.2 converges to a solution of problem (2.4).

Proof. We note that the matrix G defined by (4.8) is non-negative.
Hence from theorem 4.1, we have

$$|u_{n+1}-u_n| \leqq G\,|u_n-u_{n-1}|.$$

since $\sigma(G) < 1$, it follows that

$$\lim_{n\to\infty} |u_{n+1}-u_n| = 0 \tag{4.9}$$

Next, by inductive arguments, we have

$$\begin{aligned} |u_{n+1}-u_0| &\leqq |u_{n+1}-u_n| + \cdots\cdots + |u_1-u_0| \\ &\leqq (G^n + \cdots\cdots + I)\,|u_1-u_0| \\ &\leqq (I-G)^{-1}\,|u_1-u_0|, \end{aligned}$$

where the last inequality follows from the fact that the matrix G is nonnegative and $\sigma(G)<1$: see Ortega and Rheinboldt [27]. Hence we conclude that the sequence $\{u_{n+1}\}$ is bounded and has an accumulation point, say u^*. Let $\{u_{n_i}\}$ be a subsequence convering to u^*. Then from (4.9) we see that the sequence $\{u_{n_i+1}\}$ converges to u^* as well. Since the mappings T and g are continuous, so by passing to the limit $n_i \to \infty$, we obtain

$$u^* = u^*-g(u^*)+m(u^*)+P_K[g(u^*)-\rho E\,\{Mu^*+q+A(u^*)\} - m(u^*)],$$

which is equivalent to the general mildly (strongly) nonlinear quasi complementarity problem (2.4) by lemmas 3.1 and 3.2, that is, u^* is the solution of (2.4). We finally show that the sequence $\{u_{n+1}\}$ converges to u^*. From (4.3), we obtain

$$|u_{n+1}-u^*| \leqq G\,|u_n-u^*|,$$

where G is as defined by (4.8). Since $\sigma(G) < 1$, it follows that the entire sequence $\{u_{n+1}\}$ converges to u^*, and this completes the proof of theorem 4.2.

It can be shown using the technique of Pang [29] that condition (4.2) for the nonlinear mapping A is equivalent to the fact that A is Lipschitz continuous, that is, there exists a constant $\gamma > 0$ such that

$$\| A(u) - A(v) \| \leq \gamma \| u - v \|, \qquad \text{for all } u, v.$$

In the next theorem, we study the conditions under which the approximate solution obtained by algorithm 3.1 converges to the exact solution of the general mildly (strongly) nonlinear quasi complementarity problem (2.4). At the same time, we prove that the convergence analysis for the general mildly (strongly) nonlinear quasi complementarity problem (2.4) holds for any general closed convex set K in R^n.

Theorem 4.3. Let the mapping $T, g : R^n \to R^n$ be both strongly monotone and Lipschitz continuous respectively. If the point-to-point mapping m and the nonlinear mapping A are both Lipschitz continuous and u_{n+1} and u are solutions satisfying (3.5) and (2.4), respectively, then $u_n \to u$ strongly in R^n, for

$$\left| \rho - \frac{\alpha - \gamma(1-K)}{\beta^2 - \gamma^2} \right| < \frac{\sqrt{(\alpha - \gamma(1-K))^2 - K(2-K)(\beta^2 - \gamma^2)}}{\beta^2 - \gamma^2}$$

$$K < 1 \quad \text{and} \quad \alpha > \sqrt{K(2-K)(\beta^2-\gamma^2)} + \gamma(1-K)$$

with $K = 2(\eta + \sqrt{1 - 2\delta + \sigma^2})$,

where α, δ are the strongly monotonicity constants of T and g, and β, σ, γ and η are the Lipschitz constants of the mappings T, g, A and m respectively such that $\alpha \leq \beta$ and $\delta \leq \sigma$.

Proof. From lemmas 3.1 and 3.2, we see that the solution u of (2.4) can be characterized by the relation (3.2). Hence from (3.2) and (3.5), we have

$$\begin{aligned} \| u_{n+1} - u \| &= \| u_n - g(u_n) + m(u_n) + P_K[g(u_n) \\ &\quad - \rho(Tu_n + A(u_n)) - m(u_n)] - u + g(u) - m(u) \\ &\quad - P_K(g(u) - \rho(Tu + A(u)) - m(u)] \| \\ &\leq \| u_n - u - (g(u_n) - g(u)) \| + \| m(u_n) - m(u) \| \end{aligned}$$

$$+ \|P_K[g(u_n)-(Tu_n+A(u_n))-m(u_n)]$$
$$-P_K[g(u)-\rho(Tu+A(u))-m(u)]\|$$
$$\leqq \|u_n-u-(g(u_n)-g(u))\|+\|m(u_n)-m(u)\|$$
$$+ \|u_n-u-(g(u_n)-g(u))-(u_n-u)-\rho(Tu_n-Tu)$$
$$-\rho(A(u_n)-A(u))-(m(u_n)-m(u))\|$$

by the non-expansivity of P_K, (see lemma 3.2 in Noor [11]).

$$\leqq \|u_n-u-(g(u_n)-g(u)-g(u)\|+2\|m(u_n)-m(u)\|$$
$$+ \|u_n-u-(g(u_n)-g(u))\|+\|u_n-u-\rho(Tu_n-Tu)\|$$
$$+ \rho\|A(u_n)-A(u)\|$$

Since T, g are strongly monotone and T, g, A and m are Lipschitz continuous, so by using method of Noor [11] we get

$$\|u_n-u-(g(u_n)-g(u))\|^2 \leqq (1-2\delta+\sigma^2)\|u_n-u\| \tag{4.10}$$

and

$$\|u_n-u-\rho(Tu_n-Tu)\|^2 \leqq (1-2\alpha\rho+\beta^2\rho^2)\|u_n-u\|^2 \tag{4.11}$$

and

$$\|A(u_n)-A(u)\| \leqq \gamma\|u_n-u\| \tag{4.12}$$

and

$$\|m(u_n)-m(u)\| \leqq \eta\|u_n-u\|. \tag{4.13}$$

From (4.10), (4.11), (4.12), and (4.13), we have

$$\|u_{n+1}-u\| \leqq [2\sqrt{1-2\delta+\sigma^2}+\sqrt{1-2\alpha\rho+\beta^2\rho^2}+2\eta+\rho\gamma]\|u_n-u\|$$
$$= (K+\rho\gamma+t(\rho))\|u_n-u\|,$$

where $K = 2\sqrt{1-2\delta+\sigma^2} + 2\eta$, $t(\rho) = \sqrt{1-2\alpha\rho+\beta^2\rho^2}$.

$$\|u_{n+1}-u\| \leqq \theta\|u_n-u\|$$

with $\theta = K+\rho\gamma+t(\rho) < 1$, i.e., $K + \rho\gamma + \sqrt{1-2\alpha\rho+\beta^2\rho^2} < 1$,

from which we have

$$\left| \rho - \frac{(\alpha-\gamma(1-K)}{\beta^2-\gamma^2} \right| < \frac{\sqrt{(\alpha-\gamma(1-K)-K(2-K)\ (\beta^2-\gamma^2)}}{\beta^2-\gamma^2},$$

$$\alpha > \sqrt{K(2-K)\ (\beta^2-\gamma^2)} + \gamma(1-K), \text{ and } K < 1.$$

Since $\theta < 1$, so the fixed point problem (3.1) has a unique solution u and consequently the iterates u_{n+1} converge to u strongly in R^n, which is the required result.

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