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# On the Adjoint Situations and Reflective Subcategories

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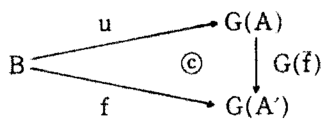
In this note, $\boldsymbol{A}$ and $\boldsymbol{B}$ are categories, the class of objects of $\boldsymbol{A}$ is denoted by obj($\boldsymbol{A}$) and the class of morphisms of $\boldsymbol{A}$ is denoted by morph($\boldsymbol{A}$).

**Definition 1.** Let $G : \boldsymbol{A} \to \boldsymbol{B}$ be a functor and let B be an object in Obj($\boldsymbol{B}$). Then (u,A), where $A \in$ obj($\boldsymbol{A}$) and $u : B \to G(A) \in$ Morph($\boldsymbol{B}$), is called a G-universal map for B if for each $A' \in$ Obj($\boldsymbol{A}$) and for each $f : B \to G(A')$ there exists a unique morphism $\bar{f} : A \to A' \in$ Morph($\boldsymbol{A}$) satisfying the commutative diagram :




**Definition 2.** Let $E : \boldsymbol{A} \to \boldsymbol{B}$ be an embedding functor.

( i ) $(r_A, A_B)$ for $B \in$ Obj($\boldsymbol{B}$) is said to be $\boldsymbol{A}$-reflection of B if $(r_A, A_B)$ is an E-universal map for B([1],[3]).

( ii ) $A$ is said to be a reflective subcategory of $\boldsymbol{B}$ if for each object $B \in$ Obj($\boldsymbol{B}$) there exists a $\boldsymbol{A}$-reflection $(r_A, A_B)$ of B ([1],[3]).

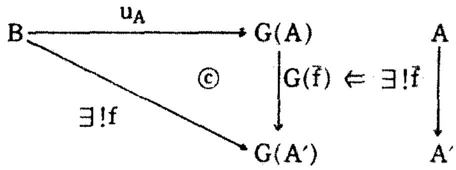
In this note we shall prove some properties with respect to universal maps (Example 3, Propositiion 4) and adjoint situation (Definition 5, Example 6). Moreover we shall prove the relation between adjoint situations and reflective subcategories (Proposition 8 and Theorem 9).

**Example 3.** Let $G : \boldsymbol{A} \to \boldsymbol{B}$ be a functor and $B \in$ Obj($\boldsymbol{B}$) be an initial object of $\boldsymbol{B}$. Then the following are equivalent.

( i ) (u,A) is a G-universal map for B where $u : B \to G(A)$ in Morph($\boldsymbol{B}$).

( ii ) A is an initial object of $\boldsymbol{A}$.

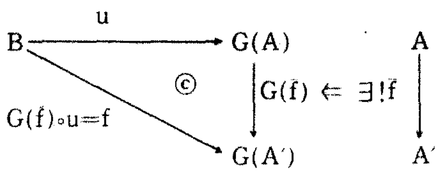
**Proof.** ( i ) ⇒ ( ii ) Since (u,A) is a G-universal map for B we have the commutative diagram:




Since B is an initial object we have only one morphism $f : B \to G(A')$ and thus $Hom(A,A')=\{\bar{f}\}$. That is A is an initial object.

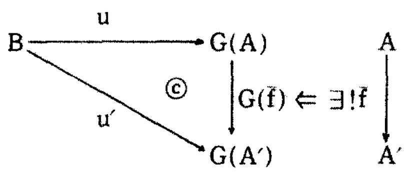
( ii ) ⇒ ( i ) Let A be an initial object of $\boldsymbol{A}$. We shall prove that (u,A) is a G-universal map for B. Since there exists only one morphism $\bar{f} : A \to A' \in Morph(\boldsymbol{A})$

$G(\bar{f}) \circ u : B \to G(A) \to G(A')$ i.e., $G(\bar{f}) \circ u : B \to G(A')$ and we have the following commutative diagram.



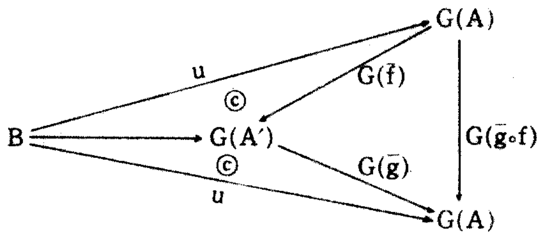

We put $f=G(\bar{f}) \circ u$ then (u,G(A)) is a G-universal map for B. / / /

**Proposition 4.** Universal maps are essentially unique. That is, if (u,A) and (u′,A′) are G-universal maps for B ($G : \boldsymbol{A} \to \boldsymbol{B}$ is a functor, $A \in Obj(\boldsymbol{A})$, $B \in Obj(\boldsymbol{B})$, and $u : B \to G(A) \in Morph(\boldsymbol{B})$ and etc.) then there is a unique isomorphism $\bar{f} : A \to A'$ such that the diagram

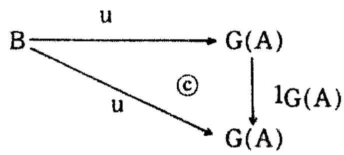



commutes.

**Proof.** We have unique morphisms $\bar{f} : A \to A'$ and $g : A' \to A$ in $Morph(\boldsymbol{A})$ satisfying the commutative diagram




On the other hand the diagram




is commutative. By the uniqueness we have $1_A = \bar{g} \circ \bar{f}$. Similarly, we can prove that $1_A = \bar{f} \circ \bar{g}$. Hence $\bar{f}$ is an isomorphism. / / /

**Definition 5.** For the functors $\boldsymbol{A} \underset{F}{\overset{G}{\rightleftarrows}} \boldsymbol{B}$, if natural transformations $\eta : 1_{\boldsymbol{B}} \to G \circ F$ and $\varepsilon : F \circ G \to 1_{\boldsymbol{A}}$ exist such that for $A \in \mathrm{Obj}(\boldsymbol{A})$

$$G(\varepsilon_A \circ \eta_{G(A)} = 1_{G(A)}$$

and for each $B \in \mathrm{Obj}(\boldsymbol{B})$

$$\varepsilon_{F(B)} \circ F(\eta_B) = 1_{F(B)}$$

then this is called anjunction or adjoint situation and is denoted by

$$(\eta, \varepsilon) : F \dashv G \text{ or simply by } F \dashv G.$$

In this case, F is said to be a left adjoint of G, G is said to be a right adjoint, $\eta$ is called the unit of the adjunction and $\varepsilon$ is called the colimit of the adjunction ([2]).

**Example 6.** Let H and K be groups which are considered as one-element categories. That is, Obj(H)={H}, Morph(H)=Hom(H,H)=the set of all endomorphism from H into itself, Obj(K)={K} and Morph(K)=Hom(K,K). Then a group epimorphism $F : H \to K$ can be regraded as a functor. It is clear that $F(1_H) = 1_K$ and for $f, g \in \mathrm{Hom}(H,H)$ $F(g \circ f) = F(g) \circ F(f)$, because of that F is a group homomorphism. Then following are equivalent :

( i ) F is an isomorphism

( ii ) F has a left adojoint

( iii ) F has a right adjoint.

**Proof.** ( i ) ⇒ ( ii ) Since F is an isomorphism we have the inverse $F^{-1} : K \to H$ such that $F^{-1}$ $^{-1}F = 1_H$ and $FF^{-1} = 1_K$.

We have the following

$$\begin{aligned} \mathrm{Hom}(H,H) &\cong \mathrm{Hom}(H,F^{-1}K) \cong \mathrm{Hom}(K,K) \\ &\cong \mathrm{Hom}(F^{-1}H \cdot K). \end{aligned}$$

Thus $F^{-1}$ is a left adjoint of F.

$$( i ) \Rightarrow ( iii ) \quad \mathrm{Hom}(K,K) \cong \mathrm{Hom}(K,F^{-1}H) \cong \mathrm{Hom}(H,H) \cong \mathrm{Hom}(F^{-1}H, K).$$

Thus $F^{-1}$ is a right adjoint of F.

In particular ( ii ) ⇔ ( iii ) is clear.

( ii ) ⇒ ( i ) Since F(H)=K by ( ii ) we have a functor $G : K \to H$ such that G(K)=H. Since

$$\begin{array}{ccc} \mathrm{Hom}(H,G(K)) & \cong & \mathrm{Hom}(F^{-1}H,K) \\ \wr\wr & & \wr\wr \\ \mathrm{Hom}(H,H) & \cong & \mathrm{Hom}(K,K) \end{array}$$

we have an isomorphism $F : H \cong K$. / / /

**Definition 7.** A morphism $f : A \to B \in \mathrm{Morph}(\boldsymbol{A})$ is said to be a section if there exists a morphism $g : B \to A$ such that $g \circ f = 1_A$.

**Proposition 8.** If (r,A) is an $\boldsymbol{A}$−reflection for B ($E : \boldsymbol{A} \to \boldsymbol{B}$ is an embedding functor) (see Definition 2).

The following are equivalent :

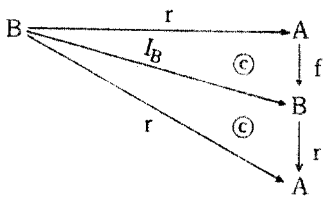
( i ) r is an isomorphism

( ii ) r is a section

**Proof.** ( i ) ⇒ ( ii ) We assume that $\boldsymbol{A}$ is a fall subcategory of $\boldsymbol{B}$. Since $r : B \to A$ is an isomorphism we have the inverse $r^{-1} : A \to B$ such that $r^{-1}r = 1_B$ and $rr^{-1} = 1_A$. By Definition 7 r is a section.

( ii ) ⇒ ( i ) Let $f : A \to B$ be the left inverse of r, i.e., $f \circ r = I_B$(since r is a section f exists).

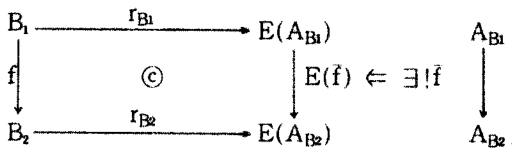
Then in the commutative diagram




we have $f \circ r = 1_A$ by the uniqueness. Hence r is an isomorphism. / / /

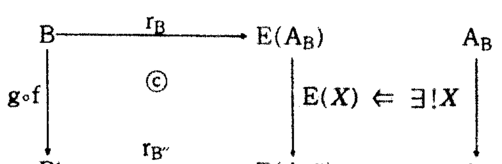
**Theorem 9.** $\boldsymbol{A}$ is a reflective subcategory of $\boldsymbol{B}$ iff the embedding functor $E : \boldsymbol{A} \to \boldsymbol{B}$ has a left adjoint $F : \boldsymbol{B} \to \boldsymbol{A}$ such that $F \dashv E$ (see Definition 5).

**Pfoof.** ( i ) We assume that $\boldsymbol{A}$ is a reflective subcategory of $\boldsymbol{B}$. Thus, for each $B \in \mathrm{Obj}(\boldsymbol{B})$ there exists an E-universal map ($r_B$, $A_B$ for $B \in \mathrm{Obj}(\boldsymbol{B})$. Let $(r_{B_i}, A_{B_i})$ $(i=1,2)$ be E-universal maps for $B_1$ and $B_2 \in \mathrm{Morph}(\boldsymbol{B})$ respectively.Then, for each morphismf : $B_1 \to B_2 \in \mathrm{Morph}\,(F\boldsymbol{B}\,)$ there exists a unique morphism $\bar{f} : A_{B1} \to A_{B2}$ satisfying the commutative diagram

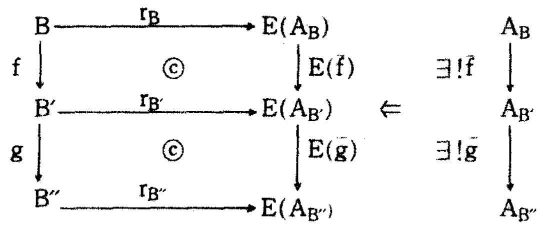



([2]). We define a functor $F : \boldsymbol{B} \to \boldsymbol{A}$ such that for each $B \in \mathrm{Obj}(\boldsymbol{B})$ $F(B)=A_B$ and for each $f : B_1 \to B_2 \in \mathrm{Morph}(\boldsymbol{B})$ $F(f) = \bar{f}$. By the uniqueness it is clear that $F(1_B) = 1_{A_B}$, and if F is a functor it is the unique one for which $F(B)=A_B$ and each square above commutes. Moreover, the commutative square above says that $r : 1_{\boldsymbol{B}} \to E \circ F$ is a natureal transformation. Thus we have to prove that $B \xrightarrow{g} B' \xrightarrow{g} B''$ in $\mathrm{Morph}(\boldsymbol{B})$ $F(g \circ f)=F(g) \circ F(f)$.

By the above reason we have a unique morphism $X : A_B \to A_B''$ satisfying the commutative square




But the above square factors as follows.

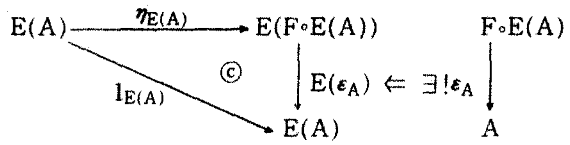



By the uniqueness $X=\bar{g}\circ\bar{f}$ and thus $F(g\circ f)=F(g)\circ F(f)$.

We shall prove that F ⊣ E, As above,

$$r_B : B — E\circ F(B)$$

is a natural transformation $\eta_B$ where $\eta : 1_{\boldsymbol{B}} \to E\circ F$. i.e., $r_B=\eta_B$. Thus for each $A \in Obj(\boldsymbol{A})$ $(\eta_{E(A)}, F\circ(E(A)))$ is an E-universal map for $E(A)$. Then we have a unique morphism $\varepsilon_A : F\circ E(A) \to A$ satisfying the commutative diagram




We can prove that $\varepsilon : 1_{\boldsymbol{A}} \to F\circ E$ is a natural transformation as follows.
For ƀ : $A \to A' \in Morph(A)$ we have

$$E(f\circ\varepsilon_A)\eta_{E(A)}= E(f)\circ E(\varepsilon_A)\circ\eta_{E(A)} =E(f)\circ 1_{E(A)}$$
$$=1_{E(A')}\circ E(f)=E(\varepsilon_{A'})\circ\eta_{E(A')}\circ E(f)$$

Since $\eta : 1_{\boldsymbol{B}} \to E\circ F$ is a natural transformation we have

$$E(f\circ\varepsilon_A)\circ\eta_{E(A)} = E(\varepsilon_A)\circ((E\circ F)f)\circ\eta_{E(A)}$$

Therefore we have $f\circ\varepsilon_A=\varepsilon_{A'}\circ(F\circ E)(f)$ ([1]) and thus $\varepsilon$ is a natural transformation.

For each $B \in Obj(\boldsymbol{B})$

$$E(\varepsilon_{F(B)}\circ F(\eta_B))\circ \eta_B=E(\varepsilon_{F(B)})\circ(E\circ F(\eta_B))\circ\eta_B.$$

Since $\eta$ is a natural transformation we have

$$E(\varepsilon_{F(B)}\circ\eta_{E\circ F(B)}\circ\eta_B=E(1_{F(B)})\circ\eta_B$$

By the property of $\eta_B$([1]) we have

$\varepsilon_{F(B)} \circ F(\eta_B) = 1_{F(B)}$.

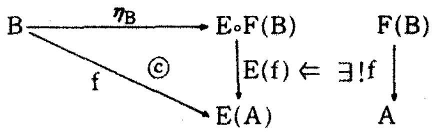
Similarly we can prove that for each $A \in Obj(\boldsymbol{A})$

$$E(\varepsilon_A) \circ \eta_{E(A)} = 1_{E(A)}$$

Therefore E has F as a left functor such that $F \dashv E$.

( ii ) We want to prove that for each $B \in Obj(\boldsymbol{B})$ $(\eta_B, F(B))$ is an E-universal map for B from the adjoint situation $(\eta, \varepsilon) : F \dashv E$.

Suppose a morphism $f : B \to E(A)$. We shall prove that there exists a unique morphism $\bar{f} : F(B) \to A$ satisfying the commutative triangle




Put $\bar{f} = \varepsilon_A \circ F(f)$ $(F(B) \xrightarrow{F(f)} F \circ E(A) \xrightarrow{\varepsilon A} A)$ then

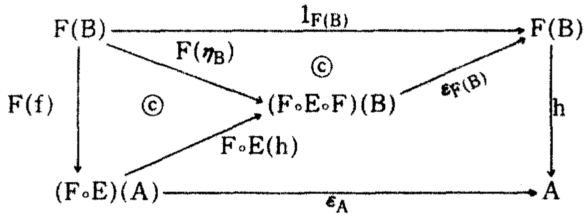
$$E(\bar{f}) \circ \eta_B = E(\varepsilon_A \circ F(f)) \circ \eta_B$$
$$= E(\varepsilon_A) \circ (E \circ F(f)) \circ \eta_B$$

Sine $\eta : 1_{\boldsymbol{B}} \to E \circ F$ is a natural transformation and $E(\varepsilon_A) \circ \eta_{E(A)} = I_{E(A)}$ we have

$$E(\varepsilon_A) \circ \eta_{E(A)} \circ f = I_{E(A)} \circ f = f$$

Hence $\bar{f}$ makes the above triange commute, because of that $E(\varepsilon_A) \circ (E \circ F)(f) \circ \eta_B = E(\varepsilon_A) \circ \eta_{E(A)} \circ f = f$.

To show uniqueness. For a morphism $h : F(B) \to A \in Morph(\boldsymbol{A})$ with $f = E(h) \circ \eta_B$. Since $\varepsilon$ is a natural transformation and $\varepsilon_{F(B)} \circ F(\eta_B) = I_{F(B)}$ we have the commutative diagram :




Therefore $h = \varepsilon_A \circ F(f)$. / / /

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