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A Note on a Fundamental Solution of the Heat Equation

Sung-Ju Kim

Department of Mathematics, Chonbuk National University,
Chonju, Chonbuk, 560—756, Korea.

As is well known, the heat equation is

$$\frac{\partial}{\partial t} E - \Delta_x E = \delta \quad \text{in } R^{n+1},$$

where R is the set of all real numbers, δ is the Dirac distribution and

$$\Delta = \Delta_x = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_n}\right)^2 .$$

In this note we shall prove that a fundamental solution of the heat equation is

$$E(x, t) = (2\sqrt{\pi t})^{-n} H(t) \exp\left(-\frac{|x|^2}{4t}\right)$$

(Lemma 1), where H(t) is the Heaviside's function and if we put

$$F(x) = \int_0^\infty E(x, t)dt$$

then we shall prove that

$$-\Delta F = \delta$$

(Theorem 3).

In other to find the fundamental solution ([1], [2]) of the equation

$$\frac{\partial E}{\partial t} - \Delta_x E = \delta \quad \text{in } R^{n+1}$$

we perform a Fourier transformation with respect to x as follows :

$$\frac{\partial \tilde{E}}{\partial t} - \sum_{i=1}^{n} e^{-i\xi_i x_i} \frac{\partial^2 E}{\partial x_i^2} dx_i = \delta(t)$$

as in [2], where $\widetilde{E}$ is a Fourier transformation of E.

Since

$$\int e^{-i\xi_i x_i}\frac{\partial^2 E}{\partial x_i^2}\,dx_i = \int i\xi_i\, e^{-i\xi_i x_i}\frac{\partial^2 E}{\partial x_i^2}\,dx_i$$
$$= \xi_i^2 \int e^{-i\xi_i x_i}\,E dx_i$$
$$= \xi_i^2\,\widetilde{E}$$

we have

$$\frac{\partial \widetilde{E}}{\partial t} + |\xi|^2\widetilde{E} = \delta(t) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\ast)$$

where $|\xi|^2 = \xi_1^2 + \cdots + \xi_n^2$. A fundamental solution of $(\ast)$ above is given by

$$\widetilde{E}(\xi, t) = H(t)\exp(-t|\xi|^2)$$

as in [1], [3] and [4].

Since $t > 0$ implies $H(t) = 1$ we take

$$\widetilde{E}(\xi, t) = \exp(-t|\xi|^2)$$

and the following lemma is proved.

Lemma 1. A fundamental solution of the heat equation is

$$E(x, t) = (2\sqrt{\pi t}\,)^{-n}H(t)\exp(-|x|^2/4t).$$

Proof. As in the above if we take

$$\widetilde{E}(\xi, t) = \exp(-t\,|\xi|^2) \quad (t > 0)$$

then

$$E(x, t) = (2\pi)^{-n}\int_{R^n}\exp(i < x, \xi > - t|\xi|^2)d\xi$$
$$= \{\prod_{j=1}^{n} (2\pi)^{-1}\int_R \exp[-t(\xi_j - i\frac{x_j}{2t})^2]d\xi_j\}$$
$$\cdot \exp(-|x|^2/4t)$$

where $< x, \xi > = x_1\xi_1 + \cdots + x_n\xi_n$.

Let z be the complex variable in the complex plane C^1 and consider the integral

$$I = \int_{\gamma} \exp(-tz^2)dz$$

where γ is any horizontal line $\operatorname{Im} z = \text{Const} = c$.
By using the Caushy's integral theorem we can compute as follows :

$$I = \int_{-\infty}^{\infty} \exp(-t(\operatorname{Re} z)^2)d(\operatorname{Re} z) = \sqrt{\frac{\pi}{t}}$$

([3]). Thus we have

$$E(x, t) = (2\sqrt{\pi t})^{-n} \exp(-|x|^2/4t).$$

Therefore, for $t \neq 0$, the inverse Fourier transformation of $\widetilde{E}(\xi, t)$ with respect to x and ξ is given by

$$E(x, t) = (2\sqrt{\pi t})^{-n} H(t) \exp(-|x|^2/4t). \qquad ///$$

Proposition 2. Let

$$E(x, t) = (2\sqrt{\pi t})^{-n} H(t) \exp(-|x|^2/4t). \qquad ///$$

be a fundamental solution of

$$\frac{\partial E}{\partial t} - \triangle_x E = \delta .$$

(i) For $t > 0$ we regard $E(x, t)$ as a distribution in the space variable, depending on the parameter t.
Then $t \to +0$ implies that $E(x, t) \to \delta(x)$.

(ii) For $n = 2$ and any test function $\varphi \in C_0^{\infty}(R^3)$ (i.e., φ is a C^{∞}– function defined on R^3 with compact support)

$$\varphi(0) = -\int_{R^3} E(x, t)(\frac{\partial \varphi}{\partial t} - \triangle_x \varphi)dxdt.$$

Proof. (i) If $x = 0$ then the heat equation becomes as

$$\partial E/\partial t = \delta .$$

Thus

$$E(0, 0) = E(0) = \int_R \delta dt = 1 = \delta(0).$$

Next, we assume that $x = (x_1, x_2, \cdots, x_n) \neq 0$.
In this case,

$$E(x, t)^2 = (2\pi)^{-n} t^{-n} H(t)^2 e^{-|x|^2/2t},$$

and since $H(t) = 1$ for $t > 0$ we have

$$E(x, t)^2 = (2\pi)^{-n} t^{-n} e^{-|x|^2/2t}.$$

We want to prove that

$$\lim_{t \to +0} E(x, t) = 0 .$$

Since

$$\lim_{t \to +0} t^{-n} e^{-\frac{|x|^2}{2t}} = 0$$

because of that

$$\begin{aligned} \lim_{t \to +0} t^{-n} e^{-\frac{|x|^2}{2t}} &= \frac{2n}{|x|^2} \lim_{t \to +0} \frac{t^{-n+3}}{e^{|x|^2/2t}} \\ &= \cdots \\ &= 0 \end{aligned}$$

we have

$$\lim_{t \to +0} E(x, t) = 0 .$$

Therefore, we get the following :

$$\lim_{t \to +0} E(x, t) = \delta(x) .$$

(ii) Since

$$\frac{\partial E}{\partial t} - \triangle_x E = \delta$$

we have the following

$$\left(\frac{\partial}{\partial t} - \triangle_x\right) E * \varphi = \delta * \varphi ,$$

where $E * \varphi$ is the convolution of E and φ .

In this case, we have the following :

$$(\frac{\partial}{\partial t} - \Delta_x)E * \varphi = < E, -(\frac{\partial}{\partial t} - \Delta_x)\varphi >$$
$$= -\int_{R^3} E(x, t)(\frac{\partial\varphi}{\partial t} - \Delta_x)\varphi \, dxdt ,$$

and

$$\delta * \varphi = < \delta, \varphi > = < \frac{d}{dx}H(x), \varphi(x) >$$
$$= < H(x), -\frac{d\varphi(x)}{dx} >$$
$$= -\int_0^\infty \varphi'(x)dx$$
$$= [-\varphi(x)]_0^\infty$$
$$= \varphi(0)$$

where $\frac{d}{dx}H(x) = \delta(x)$ ([1]). Therefore we have

$$\varphi(0) = -\int_{R^3} E(x, t)(\frac{\partial\varphi}{\partial t} - \Delta_x)\varphi \, dxdt . \qquad ///$$

Theorem 3. For $n \geqq 3$ and

$$E(x, t) = (2\sqrt{\pi t})^{-n}H(t) \exp(-|x|^2/4t)$$

we put

$$F(x) = \int_0^\infty E(x, t)dt.$$

Then

$$-\Delta F = \delta.$$

Proof. From the given expression

$$\Delta F = \int_0^\infty \Delta E(x, t)dt.$$

Since

$$\frac{\partial E}{\partial t} - \Delta_x E = \delta$$

we have

$$\triangle E = \triangle_x E = \frac{\partial E}{\partial t} - \delta .$$

Therefore

$$\begin{aligned}\triangle F &= \int_0^\infty \left(\frac{\partial E}{\partial t} - \delta\right) dt \\ &= [E(x, t)]_0^\infty - \int_0^\infty \delta(x, t)dt \\ &= -\delta(x)\end{aligned}$$

because of that

$$\lim_{t\to+0} E(x, t) = \delta(x) = 0 \quad (x \neq 0) \qquad \text{(by proposition 2)}$$

and

$$\begin{aligned}\lim_{t\to\infty} E(x, t) &= \lim_{t\to\infty} (2\sqrt{\pi t})^{-n} H(t) \exp(-|x|^2/4t) \\ &= 0 .\end{aligned}$$

Therefore we have

$$-\triangle F = \delta(x) = \delta . \qquad ///$$

References

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