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Some Bounds in a Nonlinear Eigenvalue Problem on Riemannian Manifold

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1. Introduction

Let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature and Ω be a bounded domain of M with boundary.

We consider the equation

$$(1.1)\qquad \begin{aligned} &\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda f(u) = 0 \text{ in } \Omega,\ p > 1,\ \lambda > 0 \\ &u \equiv 0 \text{ on } \partial\Omega. \end{aligned}$$

We shall only consider sufficiently smooth solutions. The set of values λ, for which (1.1) has a positive solutions, is called the spectrum.

The spectrum is an interval (λ_*, λ^*), where $0 \leqq \lambda_* < \lambda^* \leqq \infty$ and the end points λ_* and λ^* may or may not belong to the spectrum. In linear case, $f(u) = |u|^{p-2}u$ is exceptional ; then $\lambda_* = \lambda^*$. Another important distinction to be made is between the "forced case" $f(0) > 0$ and the "unforced case" $f(0) = 0$. The operator Δ_p with $p \neq 2$ arises from a variety of physical phenomena. Recently, the eigenvalue problems of Δ_p with indefinite weight with respect to Dirichlet boundary condition were investigated by Otani and Teshima [4] and Anane [1]. We refer to [2], [3] for more reference and for other aspects of Δ_p.

Sperb [6] obtained the spectrum of the equation $\Delta u + \lambda f(u) = 0$ on Euclidean space. The purpose of this paper is to show that the spectrum of the equation $\Delta_p u + \lambda f(u) = 0$, $p \geqq 2$ can be obtained on Riemannian manifold. We will use the summation convention for both kinds of indices.

2. The Auxiliary-Function for Solutions of $\Delta_p u + \lambda f(u) = 0$

Proposition 2.1. *Let M be an n-dimensional Riemannian manifold and Ω be a bounded domain of M and u be a sufficiently smooth solution of the equation*

(2.1) $$div(v(q)\nabla u) + w(q)f(u) = 0, \qquad q = |\nabla u|^2 \text{ in } \Omega.$$

If

$$G = \int_0^q \frac{v + 2v's}{w}\, ds + \alpha \int_0^u f(y)dy,$$

then

$$\Delta G + 2\frac{v'}{v} u^j u_{,k} G_{,k}^j + L_k G_{,k} \geqq (\alpha - 2)\{(v + 2qv')\frac{qf'}{v} + \frac{\alpha}{v^2}(\frac{1}{2}(v + 2qv')wf^2 - w'vf^2 q)\} + 2(\frac{v + 2qv'}{w})u^i u_{,s} R_i^s.$$

Proof. In local coordinate $(x_1, x_2, \cdots, x_n)$, the Riemannian metric is given by $g = g_{ij}dx^i dx^j$. Throughout this proof a comma followed by a subscript will denote a covarient derivative, whereas superscripts will be used for contravarient derivatives. We can see that, for each k,

(2.2) $$G_{,k} = (\frac{v + 2v'q}{w})q_{,k} + \alpha f u_{,k}\text{ , and}$$

(2.3)
$$\begin{aligned} \Delta G &= G_{,k}^{k} \\ &= (\frac{3v' + 2v''q}{w} - \frac{w'(v + 2v'q)}{w^2})\, 4u_{,l}u^{,lk}u^{,i}u_{,ik} + 2(\frac{v + 2qv'}{w})u^{,ik}u_{,ik} \\ &+ 2(\frac{v + 2qv'}{w})u^{,i}u_{,ik}{}^{,k} + \alpha f\Delta u + \alpha f'(u)q. \end{aligned}$$

where $v = v(q)$, $w = w(q)$, $f = f(u)$, and primes will denote derivatives with respect to the corresponding arguments.

In order to eleminate the term $u^{,i}u_{,ik}{}^{,k}$, we use the Ricci identity :

(2.4) $$u_{,ik}{}^{,k} = (\Delta u)_{,i} + u^{,s}R_{si}.$$

We can write (2.1) as

(2. 5)
$$\Delta u = -\frac{2v'}{v} u_{,l}u^{,li}u_{,i} - \frac{w}{v} f.$$

We differentiate (2. 5) with respect to x and multiply by u^{i}

(2. 6)
$$(\Delta u)_{,i}u^{i} = \frac{v'}{v} q_{,i}\left(\frac{2v'}{v} u_{,l}u^{,li}u_{,i} + \frac{w}{v} f\right)u^{i} - \frac{2v''}{v} q_{,i}u_{,l}u^{,lk}u_{,k}u^{i}$$
$$- \frac{2v'}{v} u_{,li}u^{,lk}u_{,k}u^{i} - \frac{2v'}{v} u_{,l}u^{,lk}{}_{,i}u_{,k}u^{i}$$
$$- \frac{2v'}{v} u_{,l}u^{,lk}u_{,ki}u^{k} - \frac{w'}{v} q_{,i}fu^{i} - \frac{w}{v} f'u_{,i}u^{i}.$$

Substituting (2. 4) into (2. 3) and using (2. 5) and (2. 6), we obtain that

(2. 7)
$$\Delta G + 2\frac{v'}{v} u^{j}u_{,k}G_{,k}{}^{j} = 2\left(\frac{v+2qv'}{w}\right)u_{,ik}u^{,ik}$$
$$+ 4u_{,i}u^{,ik}u^{j}u_{jk}\left\{\frac{3v'+2v''q}{w} - \frac{w(v+2v'q)}{w^{2}} - \frac{v'(v+2qv')}{vw}\right\}$$
$$+ 8(u_{,ik}u^{i}u_{,k})^{2}\left\{\frac{3(v')^{2}}{vw} + \frac{2qv'v''}{vw} - \frac{(2qv'+v)v'w'}{vw^{2}}\right.$$
$$\left. + \frac{(v')^{2}-vv''}{wv^{2}}(v+2qv')\right\} + 4(v+2qv')u_{,ki}u^{k}u^{i}\frac{f}{v}\left(\frac{v'}{v} - \frac{w'}{w}\right)$$
$$+ \alpha f'(u)q - 2f'q\frac{(v+2qv')}{v} - \frac{w}{v}\alpha f^{2} + 2\frac{v'}{v}\alpha f'q^{2} + 2\left(\frac{v+2qv'}{w}\right)u^{i}u_{,s}R^{s}_{i}.$$

We now apply the schwarz's inequality in the form

(2. 8)
$$u_{,ik}u^{,ik}u_{,j}u^{j} \geqq u_{,i}u^{,ik}u^{j}u_{jk}.$$

Furthermore, from (2. 2), the following identities hold

(2. 9)
$$u_{,ik}u^{i}u_{,k} = -\frac{\alpha wfq}{2(v+2qv')} + A_{k}G_{,k}$$

(2. 10)
$$(u_{,ik}u^{i}u_{,k})^{2} = \left\{\frac{\alpha wfq}{2(v+2qv')}\right\}^{2} + B_{k}G_{,k}$$

(2. 11)
$$u_{,i}u^{,ik}u^{j}u_{jk} = \left\{\frac{\alpha wf}{2(v+2qv')}\right\}^{2}q + C_{k}G_{,k}$$

where the terms A_k, B_k, C_k need not be determined explicitly. Combining (2. 7) with (2. 8)~(2. 11), we are led to

$$\Delta G + 2\frac{v'}{v} u_{,k}u^{j}G_{,j}^{k} + L_kG_{,k} \geq (\alpha - 2)\{(v + 2qv')\frac{qf'}{v}$$
$$+ \frac{\alpha}{v^2}(\frac{wf^2}{2}(v + 2qv') - w'vf^2q)\} + 2(\frac{v + 2qv'}{w})u^{i}u_{,s}R_i^s.$$

where the term L_k is singular at the point where $\nabla u = 0$.

Lemma 2.2. *Let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature. Let Ω be a bounded domain of M and the mean curvature of $\partial\Omega$ be positive. Let f be an increasing function such that $f(s) \geq 0$, for $s > 0$ and let u be a solution of the equation*

(2. 12)
$$div(q^{\frac{p-2}{2}}\nabla u) + \lambda f(u) = 0 \ in\ \Omega,\ q = |\nabla u|^2,\ \lambda > 0,\ p > 0,$$
$$u \equiv 0 \quad on \quad \partial\Omega.$$

then

$$G = \frac{1}{\lambda}\frac{2(p-1)}{p} q^{\frac{p}{2}} + 2\int_0^u f(y)dy$$

has a maximum at $\nabla u = 0$.

Proof. By Proposition 2.1 and maximum principle [5], G has a maximum at $\nabla u = 0$ or at boundary point of Ω. We assume that G has a maximum point x_0 on $\partial\Omega$. We may choose an orthonormal frame field $e_1, e_2, \cdots, e_n = \frac{\partial}{\partial\nu}$ at x_0 where ν is the unit outward normal vector. Then

$$0 \leq \frac{\partial G}{\partial\nu}(x_0) = 2\cdot\frac{p-1}{\lambda} q^{\frac{p}{2}-1}\frac{\partial u}{\partial\nu}\frac{\partial^2 u}{\partial\nu^2} + 2f(u)\frac{\partial u}{\partial\nu}.$$

In addition on $\partial\Omega$, (2. 12) can be written as

(2. 13)
$$(p-2)q^{\frac{p-4}{2}}(\frac{\partial u}{\partial\nu})^2(\frac{\partial^2 u}{\partial\nu^2}) + q^{\frac{p-2}{2}}\Delta u + \lambda f(u) = 0.$$

Let M_0 be a mean curvature at x_0. It is well known that, [6]

(2. 14)
$$\Delta u = \Delta_{\partial\Omega}u + (n-1)M_0\frac{\partial u}{\partial\nu} + \frac{\partial^2 u}{\partial\nu^2} = (n-1)M_0\frac{\partial u}{\partial\nu} + \frac{\partial^2 u}{\partial\nu^2}$$

at x_0. Substituting (2. 14) into (2. 13), we obtain

$$\frac{\partial^2 u}{\partial\nu^2} = \frac{-(n-1)M_0}{(p-1)}\frac{\partial u}{\partial\nu} - \frac{\lambda f(u)}{(p-1)}q^{-(\frac{p-2}{2})}.$$

Hence it holds that

$$\frac{\partial G}{\partial \nu}(x_0) = -\frac{2(n-1)}{\lambda} M_0 q^{\frac{p}{2}} \leq 0.$$

Therefore we have, $\frac{\partial G}{\partial \nu}(x_0) = 0$. If $\frac{\partial G}{\partial \nu}(x_0) = 0$ then $q = 0$ at x_0. Hence G has a maximum at $\nabla u = 0$.

3. Main Theorems

Theorem 3.1. *Let M be an $n-$dimensional Riemannian manifold with non$-$negative Ricci curvature. Let Ω be a bounded domain of M and the mean curvature of $\partial\Omega$ be positive. Suppose that*

(a) $\lim_{s \to 0} \frac{s^{p-1}}{f(s)} = l_0$, $\lim_{s \to \infty} \frac{s^{p-1}}{f(s)} = l_\infty > 0$,

and f is a positive increasing function for $s > 0$, $f(0) = 0$,

(b) $\inf_{s>0} H(s) = H_0$ *with* $H(s) = \int_0^s (F(s) - F(t))^{-\frac{1}{p+1}} dt$, $\frac{dF}{ds} = f(s)$.

If u is a positive solution of the equation (2. 12) then

$$\lambda \geq \lambda_* \geq \frac{p-1}{p}\left(\frac{H_0}{d}\right)^p > 0$$

where d is the radius of the largest geodesic ball contained in Ω.

Proof. We define a function $G(x) = \frac{2}{\lambda}\frac{p-1}{p} q^{\frac{p}{2}} + 2\int_0^u f(y)dy$ where $q = |\nabla u|^2$. By Lemma 2.2, G has a maximum at $\nabla u = 0$. Then

$$G(x) = \frac{2(p-1)}{\lambda p} q^{\frac{p}{2}} + 2\int_0^u f(y)dy \leq 2F(u_m)$$

where $u_m = \max_{x \in \Omega} u(x)$. This inequality can be written as

$$\frac{|\nabla u|}{(F(u_m) - F(u))^{1/p}} \leq \left(\frac{\lambda p}{p-1}\right)^{\frac{1}{p}}. \tag{2. 15}$$

Let x_0 be the point where $u = u_m$ and $\bar{x}$ a point on $\partial\Omega$ nearest to x_0 in geodesic distance. Let γ be a geodesic joining a point x_0 to a point $\bar{x}$ and parametrized by arc length. Since

$$\int_{\gamma}|\nabla|ds \geqq \int_{\gamma}\nabla u \cdot \frac{d\gamma}{ds}ds = \int_{\gamma}\frac{d(u(\gamma(s)))}{ds}ds = \int_{0}^{u_m}du,$$

we have

$$H(u_m) = \int_{0}^{u_m}\frac{du}{(F(u_m)-F(u))^{1/p}} \leqq (\lambda\frac{p}{p-1})^{\frac{1}{p}}d.$$

In particular, it follows that $H_0 = \inf_{s>0}H(s) \leqq (\frac{\lambda p}{p-1})^{\frac{1}{p}}d$. Since f is increasing, we have for $s > t$, $F(s) - F(t) \leqq f(s)(s-t)$ so that

$$H(s) = \int_{0}^{s}\frac{dt}{(F(s)-F(t))^{1/p}} \geqq \frac{p}{p-1}(\frac{s^{p-1}}{f(s)})^{\frac{1}{p}} > 0.$$

Because of our assumption in (a), we see that $H_0 > 0$, and clearly we must have $\lambda \geqq \lambda_* \geqq (\frac{p-1}{p})(\frac{H_0}{d})^p > 0$.

Corollary 3.2. *Let M and Ω satisfy the assumptions of Theorem 3.1 and u be a solution of the equation (2. 12) with f is an increasing function for $s > 0$ and $f(0) > 0$. Then $\lambda \geqq \lambda_* = 0$.*

Proof. In proof of Theorem 3.1,

$$H(s) = \int_{0}^{s}\frac{dt}{(F(s)-F(t))^{1/p}} \geqq \frac{p}{p-1}(\frac{s^{p-1}}{f(s)})^{\frac{1}{p}} \geqq 0.$$

Hence $H_0 = \inf_{s>0}H(s) = 0$ and $\lambda \geqq 0$.

Theorem 3.3. *Let M be an n-dimensional Riemannian manifold with nonnegative Ricci curvature. Let Ω be a bounded domain of M and the mean curvature of $\partial\Omega$ be positive. Let f be an increasing function and $f(s) > 0$ for $s \geqq 0$.*
Then equation (2. 12) has a positive solution for $\lambda \in (0, \lambda^*)$ and

$$\lambda^* \geqq (\frac{p}{p-1})^{p-1}d^{-p}\sup_{s>0}\frac{s^{p-1}}{f(s)}$$

where d is the largest radius of geodesic ball contained in Ω.

Proof. If $u_- \equiv 0$, then, in (2. 12), $\lambda f(u) \geqq 0$. Hence u_- is a subsolution of (2. 12). Let ψ be a solution of the equation

$$\mathrm{div}(|\nabla\psi|^{p-2}\nabla\psi) + 1 = 0 \quad \text{in} \quad \Omega, \ \psi \equiv 0 \quad \text{on} \quad \partial\Omega.$$

Define $\overline{u}$ by $c\psi$, where c is constant. Let ψ_m be the maximum value of $\psi(x)$ in Ω. Since

$$\mathrm{div}(|\nabla\overline{u}|^{p-2}\nabla\overline{u}) + \lambda f(\overline{u}) = \mathrm{div}(c^{p-1}|\nabla\psi|^{p-2}\nabla\psi) + \lambda f(c\psi)$$

If $-c^{p-1} + \lambda f(c\psi_m) < 0$, then $\overline{u}$ is a super solution of (2. 12). Hence, for $\lambda \leqq \frac{c^{p-1}}{f(c\psi_m)}$, the equation (2. 12) has a solution. Using Proposition 2.1 we see that $\psi_m \leqq \frac{p-1}{p} d^{\frac{p}{p-1}}$. Hence it holds that

$$\frac{c^{p-1}}{f(c\psi_m)} \geqq \frac{c^{p-1}}{f(c\frac{p-1}{p} d^{\frac{p}{p-1}})} .$$

Let $s = c\frac{p-1}{p} d^{\frac{p}{p-1}}$. Then

$$\frac{c^{p-1}}{f(c\psi_m)} \geqq \frac{s^{p-1}(\frac{p}{p-1})^{p-1}d^{-p}}{f(s)} .$$

In the case of

$$\lambda \leqq (\frac{p}{p-1})^{p-1}d^{-p} \sup_{s>0} \frac{s^{p-1}}{f(s)} ,$$

the equation (2. 12) has a positive solution and

$$\lambda^* \geqq (\frac{p}{p-1})^{p-1}d^{-p} \sup_{s>0} \frac{s^{p-1}}{f(s)} .$$

Remark. In case that M is an n-dimensional Euclidean space and $p = 2$, Sperb [6] obtained that $\lambda^* \geqq \frac{8n}{d^2} \sup_{t>0}\frac{t}{f(t)}$.

Corollary 3.4. *Let M, Ω, H, and f satisfy the hypothesis of theorem 3.1. Then the equation (2. 12) has a positive solution for $\lambda \in (\lambda_*, \lambda^*)$*

where $\lambda_* \geqq \frac{p-1}{p} \ (\frac{H_\Omega}{d})^p > 0$ and $\lambda^* \geqq (\frac{p}{p-1})^{p-1}d^{-p} \sup_{s>0}\frac{s^{p-1}}{f(s)}$.

Proof. By the same method that we used in theorem 3.3, we obtain that

$$\lambda^* \geq \left(\frac{p}{p-1}\right)^{p-1} d^{-p} \sup_{s>0} \frac{s^{p-1}}{f(s)} .$$

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