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# A Note on a Compact Immersion

Jae Up So

*Department of Mathematics, Chonbuk National University, 560–756, Chonju, Korea*

## 1. Introduction and Notation

Let $N$ be a Riemannian manifold with sectional curvatures $K_N \geqq b$. Let $B_r$ be a closed normal ball of radius $r$ in $N$, and if $b>0$, assume that $r < \frac{\pi}{2\sqrt{b}}$. If $N$ is a space from $\tilde{S}^n_b$ of constant sectional curvature $b$, then the mean curvature vector $H(\partial B_r)$ of the geodesic hypersphere $\partial B_r$ in N is known to have the length $\|H(\partial B_r)\| = h_b(r)$, where (c.f. [1])

$$h_b(r) = \begin{cases} \sqrt{b}\cot(\sqrt{b}\, r & \text{if } b>0\,; \\ \frac{1}{r} & \text{if } b=0\,; \\ \sqrt{-b}\coth\sqrt{-b}\, r) & \text{if } b<0. \end{cases}$$

In [2], Markvorsen proved

**Theorem 1** (Markvorsen). If $K_N \leqq b$ and if $\|H(\partial B_r)\| = h_b(r)$ at every point of $\partial B_r$ then the interior of $B_r$ is isometric to a ball of radius $r$ in $\tilde{S}^n_b$, and $\partial B_r$ is therefore isometric to $(S^{n-1}, g_b(r)$ with constant curvature $K(b,r) = (h_b(r))^2 + b$.

The main object of this paper is to show a result, which is complementary to the above Theorem 1.

**Theorem 2.** If $K_N \geqq b$ and if $\|H(\partial B_r)\| = h_b(r)$ at every point of $\partial B_r$, then the interior $B_r$ is isometric to a ball of radius $r$ in $\tilde{S}^n_b$, and $\partial B_r$ is therefore isometric to $(S^{n-1}, g_b(r))$ with constant curvature $K(b,\ r) = (h_b(r))^2 + b$.

## 2. Preliminary Results

Let $M$ be a compact, connected Riemannian manifold and let $\Phi : M \to B_r \subset N$ be an isometric immersion into the Riemannian manifold $N$ with sectional curvature $K_N \geqq b$, such that $\Phi$ has its image in a closed normal ball $B_r$ of radius $r$ with $r < \frac{\pi}{2\sqrt{b}}$ if $b>0$.

Let $p$ be the center of the normal ball $B_r$ in $N$, and consider the unique normal geodesic $\gamma : [0, l] \to B_r$ from $\gamma(0) = p$ to $\gamma(l) = q \in M$(simplifying notations we write $M$ for $\Phi(M) \subset B_r$).

Let $X \in T_qM$ be a unit vector and consider the unique Jacobi field $V$ along $\gamma$ with $V(0) = 0$ and $V(l) = X$. Let $I(Y,Z)$ denote the index form along $\gamma$. Since $K_N \geqq b$, we obtain

**Lemma 2.1**

$$I(V,V) \geqq \frac{1}{l} (\langle X, \gamma' (l)\rangle^2 + h_b(l)). \tag{2.1}$$

**Proof.** In the space form $\tilde{S}^n_b$ of constant curvature $b$, let $\sigma$ be a normal geodesic. Let $\{X_i\}_{i=1, \cdots, n-1}$ be an orthonormal frame on $\gamma$, and let $\{Y_i\}_{i=1, \cdots, n-1}$ be an orthonormal frame on $\sigma$.

If $X = \sum_{i=1}^{n-1} a_i X_i(l) + a_n\dot{\gamma}(l)$, then take a vector $\tilde{X} = \sum_{i=1}^{n-1} a_i Y_i(l) + a_n\dot{\sigma}(l)$, on $\sigma$.

Let $\tilde{W}$ be the unique Jacobi field along $\sigma$ with $\tilde{W}(0) = 0$ and $\tilde{W}(l) = \tilde{X}$, and let W be a vector field along $\gamma$ with the same expression in the parellel frames $\{X_i\}$, $\{Y_i\}$ as $\tilde{W}$. That is, if $\tilde{W}(t) = \sum_{i=1}^{n-1} a_i(t) Y_i(t) + a_n(t)\sigma'(t)$, then $W(t) = \sum_{i=1}^{n-1} a_i(t) X_i(t) + a_n(t)\gamma'(t)$.

We then obtain

$$I(V,V) \leqq I(W,W) \leqq \tilde{I}(\tilde{W}, \tilde{W}) \tag{2.2}$$

To calculate $\tilde{I}(\tilde{W}, \tilde{W})$, we put $T(t) = \dot{\sigma}(t)$, and define a parellel field $Y(t)$ along $\sigma$ by requiring

$$Y(l) = \tilde{W}(l) - \langle \tilde{W}(l), T(l)\rangle T(l).$$

Note that $Y(t) \perp T(t)$ for all t.

Being a Jacobi field in a space form, $\tilde{W}$ can be decomposed uniquely into

$$\tilde{W}(t) = ktT(t) + \phi(t)Y(t),$$

where $k = \frac{1}{l} \langle \tilde{W}(l), T(l) \rangle$, and where $\phi(t)$ satisfies the differential equation

$$\phi''(t) + b\phi(t) = 0,$$

with the boundary conditions $\phi(0) = 0$ and $\phi(l) = 1$. Here, we have

$$\psi'(l) = \begin{cases} \sqrt{b}\, cot(\sqrt{b}\, l) & \text{if } b > 0\,; \\ \frac{1}{l} & \text{if } b = 0\,; \\ \sqrt{-b}\, coth\,(\sqrt{-b}\, l) & \text{if } b < 0. \end{cases}$$

And so, $\psi'(l) = h_b(l)$.

Using this information and the fact that $\tilde{I}\ (\tilde{W}, \tilde{W}) = <\tilde{W}', \tilde{W}>|_{t=l}$, we get

$$\tilde{I}(\tilde{W}, \tilde{W}) = \frac{1}{l} < \tilde{W}(l), T(l) >^2 + h_b(l) \,||\, Y(l) \,||^2,$$

where

$$||\, Y(l)\,||^2 = ||\, \tilde{W}(l)\,||^2 - <\tilde{W}(l), \quad T(l)>^2.$$

The lemma now follows from (2.2), since

$$||\tilde{W}(l)\,||^2 = ||X||^2 = 1,$$

and

$$<\tilde{W}(l), T(l)>^2 = < X, \gamma'(l) >^2.$$

Before proving Theorem 2, we consider again $\Phi(M) = M \in B_r(p)$. Define the function $f : M \to \mathbf{R}$ by

$$f(q) = \frac{1}{2} d\,(p,q)^2$$

for $q \in M$.

**Lemma 2.2**

With the notation of the lemma 2.1.

$$\text{Hessian of } f(q)(X,X) = \nabla^2 f(q)(X,X) = l(I(V,V) + \langle \alpha(X,X), \dot{\gamma}(l)\rangle),$$

where $\alpha$ is the second fundamental form of $M$ at $q$ in $N$.

**Proof.**

$$\begin{aligned} Hess.f(X,Y) &= D(Df)(X,Y) \\ &= D(df)(X,Y) \\ &= (\nabla_Y df)(X) \\ &= Y(df(X)) - df(\nabla_Y X) \\ &= YXf - (\nabla_Y X)(f) \\ &= XYf - (\nabla_X Y)(f) \\ &= X\langle \text{grad} f, Y\rangle - \langle \text{grad} f, \nabla_X Y\rangle \\ &= \nabla_X(\text{grad} f), Y\rangle \end{aligned}$$

where $\nabla$ is the Riemannian connection on M.

Define the function $g : N \to \mathbf{R}$ by $g(q) = \frac{1}{2}d(p,q)^2$ for $q \in N$. Then, $g \mid M = f$ and

$$\begin{aligned} Hess.f(X,Y) &= XYg - (\overline{\nabla}_X Y)(g) \\ &= XYf - ((\overline{\nabla}_X Y)^T f + (\overline{\nabla}_X Y)^{\perp}(g)) \\ &= Hess.f(X,Y) - (\overline{\nabla}_X Y)^{\perp}(g) \\ &= Hess.f(X,Y) - \langle \text{grad } g, (\overline{\nabla}_X Y)^{\perp}\rangle \\ &= Hess.f(X,Y) - \langle \alpha(X,Y), \text{grad } g\rangle, \end{aligned}$$

where $\overline{\nabla}$ is the Riemannian connection on N.

Since $g$ has constant value on $\partial B_t$, grad $g$ is orthogonal to $\partial B_t$, and grad $g = a(t)\dot{\gamma}(t)$.

Hence, $\langle \text{grad } g, \dot{\gamma}(t)\rangle = \dot{\gamma}(t)(g) = \frac{dg}{dt}(t) = t$, and so grad $g = t\dot{\gamma}(t)$. Thus

$$\begin{aligned} Hess.f(q)(X,X) &= Hess.g(q)(X,X) + \alpha(X,X), \text{grad } g\rangle_q \\ &= \langle \overline{\nabla}_X \text{grad } g, X\rangle_q + \langle \alpha(X,X), l\dot{\gamma}(l)\rangle \\ &= l\langle \overline{\nabla}_V \dot{\gamma}(l), X\rangle_q + l\langle \alpha(X,X), \dot{\gamma}(l)\rangle \\ &= l\langle \overline{\nabla}_{\dot{\gamma}(l)} V, X\rangle_q + l\langle \alpha(X,X), \dot{\gamma}\rangle \\ &= l\langle V', V\rangle + l\langle \alpha(X,X), \dot{\gamma}(l)\rangle \\ &= l(I(V,V) + \langle \alpha(X,X), \dot{\gamma}(l)\rangle) \end{aligned}$$

## 3. Proof of Theorem 2

We let $M = \partial B_r$ and use Lemma 2.2. Since $f$ has constant value on $\partial B_r$, we get

$$-\langle \alpha(X,X), \gamma'(r) \rangle = I(V,V). \tag{2.3}$$

Hence, from Lemma 2.1 and the fact that $\langle X, \gamma'(r) \rangle = 0$, we obtain

$$-\langle \alpha(X,X), \gamma'(r) \rangle \leqq h_b(r). \tag{2.4}$$

Since $h_b(r) = \|H_q(\partial B_r)\| = \frac{1}{n-1} \Sigma_{i=1}^{n-1}(-\langle \alpha(X_i,X_i), \dot{\gamma}(r) \rangle)$, where $\{X_i\}$ is an orthonormal basis of $T_q(\partial B_r)$, we must have equality in (2.4) for every X. But, then, from (2.3),

$$I(V,V) = \tilde{I}(\tilde{W}, \tilde{W}) = h_b(r).$$

Since $K_N \geqq b$, this is only possible if $K_N(V \wedge \dot{\gamma}(t)) \equiv b$. This is then for every $\gamma$ connecting p and $\partial B_r$ and for every orthogonal Jacobi field V along $\gamma$.

Let $i : T_pM \to T_{\tilde{p}}\tilde{S}^n_b$ be a linear isometry. Let $\exp_p^{-1}(q) = x$ and let $\gamma_x$ is the unique geodesic in $B_r$ such that $\gamma_x(0) = p$ and $\dot{\gamma}_x(0) = 0$. Defien a map $\phi_i$ from the interior of $B_r$ to the interior of $S_r$ by

$$\phi_i(q) = \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q),$$

where $S_r$ is the ball of radius r in $\tilde{S}^n_b$ with center $\tilde{p}$.

To show $\phi_i$ is an isometry, it is enough to show that $\|d\phi_i(v)\| = \|v\|$, for all q in the interior of $B_r$ and for all $v \in T_qM$ with $v \perp \dot{\gamma}x(1)$. Because

$$\|d\phi_i(\gamma'_x(1))\| = \|\gamma'_{ix}(1)\| = \|ix\| = \|x\| = \|\gamma'_x(1)\|.$$

Let $d\exp_p(y_x) = v$, where $v \in T_qM$ with $v \perp \gamma'_x(1)$. Then, $d\exp_p(y_x) = V(1)$, where V is the unique orthogonal Jacobi field along $\gamma_x$ such that $V(0) = 0$ and $V'(0) = y \in T_pM$(c.f.[3]). Since i is linear, $di(y_x) = (iy)_{ix}$ and so,

$$d\phi_i(v) = d\exp_{\tilde{p}}((iy)_{ix}).$$

Thus

$$d\phi_i(v) = d\exp_{\tilde{p}}((iy)_{ix}) = \tilde{V}(1),$$

where $\tilde{V}$ is the unique orthogonal Jacobi field along the geodesic $\gamma_{ix}$ in $S_r$ such that $\tilde{V}(0) = 0$ and $\tilde{V}'(0) = iy \in T_{\tilde{p}}\tilde{S}^n_b$.

From the Rauch Comparison Theorem, $\|V(1)\| = \|\tilde{V}(1)\|$. Hence

$$\|d\phi_i(v)\| = \|\tilde{V}(1)\| = \|V(1)\| = \|v\|.$$

We conclude that the interior of $B_r$ is isometric to a geodesic ball of radius $r$ in $\tilde{S}^n_b$. Finally, the Gauss equation in connection with

$$\langle \alpha(X_i, X_i), \alpha(X_j, X_j) \rangle = (h_b(r))^2$$

shows that $\partial B_r$ has the constant curvature mentioned in Theorem 2.

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