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Unipotent Subgroups of Orthogonal and Symplectic Groups Associated with Nonreduced Root Systems of Type BC_n

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1. Introduction

The set of roots of a semisimple Lie algebra over the complex number field with respect to a Cartan subalgebra forms a reduced root system and some simple groups like Chevalley groups are obtainable as groups of automorphisms of such Lie algebra. There are irreducible reduced root systems of 9 different types. But the irreducible nonreduced root systems are precisely of type BC_n, which is the union of reduced root systems of type B_n and C_n. We obtain this nonreduced root system from a reduced root system by omitting the condition that the only multiples $\pm\alpha$ of each root α can be roots again (cf. [4], [5]).

Any unipotent subgroup of general linear groups can be associated with reduced root systems of type A_n, and the reduced root systems of type D_n, B_n, and C_n describe generators and relations of the maximal unipotent subgroups of the even orthogonal, odd orthogonal, and sympletic groups, respectively (cf. [2, 9, and 10]).

In this paper, by describing generators and relations, we associate the positive nonreduced root systems of type BC_n with unipotent radicals of certain parabolic subgroups of orthogonal and symplectic groups over all fields of characteristic other than 2. This association seems strange but even the reduced root systems of type C_n can be associated with some unipotent subgroups of even orthogonal gorups (cf. [8]). The author did the analogous work roughly for symplectic groups in her thesis (cf. [7]).

In Section 2, we introduce some notations and terminologies related to the abstract nonreduced root system. In Section 3, we use block matrix notations to define suitable unipotent subgroups of orthogonal and symplectic groups which will be associated with nonreduced root systems of type BC_n in the section 4. Many notations for this paper are newly created based on [1, 2, 3, and 9].

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2. The Abstract Nonreduced Root System

We follow all notations from [8] except we use the minus notation, $-i$, as in [3], instead of the underlining or overlining notation ($\underline{i}$ or $\bar{i}$) used in [6, 7, and 8], since $-i$ seems more reasonable than others. By the $(-i, -j)$ entry of an $m \times n$ matrix M we denote the $(m-i+1, n-j+1)$ entry of M. In order words, we may consider each row [column] number $-i$ [$-j$] as a positive number $(m-i+1)$ [$(n-j+1)$] between 1 and m [n]. That is, the $(-k)$-th row [column] is the k-th row [column] counted from the last row [column]. However, this minus notation is, unfortunately, different from that of [1], where $-i$ means $m+i$ as a positive number between 1 and $2m$ for $1 \leqq i \leqq m$.

Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the usual orthonormal vectors for an n-dimensional real euclidean space. Then the positive nonreduced root system of type BC_n consists of four different types of roots denoted by

$$R_{i,j} = \varepsilon_i - \varepsilon_j, \quad R_{i,-j} = \varepsilon_i + \varepsilon_j, \quad R_{i,-i} = 2\varepsilon_i, \text{ and } R_{i,n+1} = \varepsilon_i,$$

and $R_{i,n+1} = R_i$ are fundamental roots, where $1 \leqq i < j \leqq n$, $1 \leqq i \leqq n$. Note that

$$R_{i,n+1} = R_i + R_{i+1} + \cdots + R_n \text{ and } R_{i,-i} = 2R_{i,n+1}.$$

The nonreduced root system has three different root lengths, unlike the reduced root systems having at most two different root lengths. So we may call roots $R_{i,j}$ and $R_{i,-j}$ of lengh $\sqrt{2}$ as ***medium roots***, $R_{i,-i}$ of length 2 as ***long roots***, and $R_{i,n+1}$ of length 1 as ***short roots***. Then the medium roots of type BC_n are also the short roots of type C_n and long roots of type B_n. We have an analogous notion of the height $ht(R)$ of a root R in the positive nonreduced root system of type BC_n, and roots are ordered as usual so that $ht(R) \leqq ht(S)$ whenever $R < S$.

Since we use R_n as the fundamental short root $R_{n,n+1}$ in this paper, when we use the results of [6, 8], we must always write the fundamental long root R_n in [6, 8] as $R_{n,-n}$ in an unabbriviated notation and interpret it as a long root of height 2. In [6, 8], where we associated some nonmaximal unipotent subgroup with the reduced root system of type C_n, the fundamental long root R_n has always been identified with $R_{n,-n}$ instead of $R_{n,n+1}$, even though the two subscripts $-n$ and $n+1$ were identical.

3. Some unipotent subgroups U of orthogonal and symplectic groups

Let K be a field of characteristic not equal to 2 and let J be a square matrix with 1 on the minor diagonal and 0 elsewhere, i. e.,

$$J = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot & & \\ 1 & & & \end{bmatrix} = \sum_i e_{i,-i},$$

Let V be a $2l$ dimensional [$(2l+1)$ dimensional] vector space over K. If we consider J as the matrix representation of a nonsingular symmetric bilinear form on V with respect to a suitable basis, then the orthogonal group $G = O(2l, K)$ [$G = O(2l+1, K)$] consists of all nonsingular matrices A such that ${}^{T}AJA = J$. We denote by $O(K)$ either $O(2l, K)$ or $O(2l+1, K)$.

Let V be a $2l$ dimensional vector space over K. If we consdier $\mathcal{J}$ as the matrix representation of a nondegenerate alternating bilinear form on V with respect to a suitable basis, where

$$\mathcal{J} = \begin{bmatrix} 0 & J \\ -J & 0 \end{bmatrix},$$

then the symplectic group $G = S_p(2l, K)$ consists of all nonsingular matrices A such that ${}^{T}A\mathcal{J}A = \mathcal{J}$.

Now we define suitable (nonmaximal) unipotent subgroups U of orthogonal and symplectic groups G which will be associated with nonreduced root systems of type BC_n in the next section. Let F be a flag in V which is a chain

$$0 = V_0 \subset V_1 \subset \cdots \subset V_i \subset \cdots \subset V_n$$

of totally isotropic subspaces of V, and assume that F satisfies the following conditions in each case :

(1) Case when $G = S_p(2l, K)$, $l \geqq 3$: $\dim V_n < l$ in order to get nontrivial elements corresponding to all short roots.

(2) Case when $G = O(2l, K)$, $l \geqq 5$: $\dim V_n < l$ and $\dim(V_i / V_{i-1}) \geqq 2$ for all $i = 1, 2, \cdots, n$ in

order to get nontrivial elements corresponding to all short and long roots.

(3) Case when $G=O(2l+1,K)$, $l\geqq 4$: $dim(V_i/V_{i-1})\geqq 2$ for all $i=1,2,\cdots,n$. But it does not matter whether $dim\,V_n$ is equal to l or not.

Then the stabilizer of a flag F is a parabolic subgroup P of G and P admits the Levi decomposition $P=L\cdot U$ where L is the Levi subgroup of P and U is the unipotent radical of P.

If F is the maximal flag with no restriction, then the corresponding parabolic subgroups are Borel subgroups, and the positive reduced root systems of type D_l, B_l, and C_l are associated with maximal unipotent subgroups of $O(2l,K)$, $O(2l+1,K)$, and $S_p(2l,K)$, respectively (cf. [2], [10]).

Let A be any element contained in a unipotent subgroup U defined as above. Then A is a $(2n+1)\times(2n+1)$ block matrix written as

$$I+\sum_{s<t}A_{s,t}E_{s,t}$$

($A_{s,t}E_{s,t}$ denoting the matrix $A_{s,t}$ in the (s,t) block), which is upper triangular and whose diagonal blocks are the identity matrices, and A looks like the following block matrix :

$$\begin{array}{cccc|c|cccc}
1 & 2 & \cdots & n & n+1 & -n & \cdots & -2 & -1 \\
I & A_{1,2} & \cdots & A_{1,n} & A_{1,n+1} & A_{1,-n} & \cdots & A_{1,-2} & A_{1,-1} \\
 & I & \cdots & A_{2,n} & A_{2,n+1} & A_{2,-n} & \cdots & A_{1,-2} & A_{2,-1} \\
 & & & & & \vdots & & & \\
 & & & I & A_{n,n+1} & A_{n,-n} & \cdots & A_{n,-2} & A_{n,-1} \\
\hline
 & & & & I & A_{n+1,-n} & \cdots & A_{n+1,-2} & A_{n+1,-1} \\
\hline
 & & & & & I & \cdots & A_{-n,-2} & A_{-n,-1} \\
 & & & & & & & \vdots & \\
 & & & & & & & I & A_{-2,-1} \\
 & & & & & & & & I
\end{array}$$

whence comparing with the case of [6, 8], we now have just one more column block and row block in the center which are the $(n+1)$-th blocks, and these blocks will be related to short roots in the next section. Gathering some block entries together, we can express A as

$$I+\sum_{1\le i<j\le n}\{(A_{i,j}E_{i,j}+A_{-j,-i}E_{-j,-i})+(A_{i,-j}E_{i,-j}+A_{j,-i}E_{j,-i})\}$$
$$+\sum_{1\le i\le n}\{(A_{i,n+1}E_{i,n+1}+A_{n+1,-i}E_{n+1,-i})+A_{i,-i}E_{i,-i}\},$$

where each (i,j) block entry $A_{i,j}$ is an $m_i\times n_{j-1}$ matrix over K with $m_i=n_{i-1}=dim(V_i/V_{i-1})$ for each $1\leqq i,j\leqq n$ and the sizes of all the other block entries are determined accordingly.

Note the following hold :

(i) $n+1 = -(n+1)$ as a row(column) number between 1 and $2n+1$.

(ii) $n_n = m_{n+1} = dimV-dimV_n$ and n_n is even($\geqq 2$) if $G = S_p(2l,K)$, $O(2l,K)$, and odd($\geqq 1$) if $G = O(2l+1, K)$.

(iii) $m_i\geqq 2$ for all i, if $G = O(K)$.

Notations. While ${}^TA$ is the ordinary transpose of a matrix A with respect to the main diagonal, by ${}^JA$ we denote the ***transpse of A with respect to the minor diagonal.*** Let J and $\mathcal{J}$ be square matrices of suitable sizes of the same form as those given in the above. In fact, ${}^JA$ is just J^TAJ. If $A = -{}^JA$, then A is a skew symmetric with respect to the minor diagonal and has all zeroes in the minor diagonal entries. Also, for symplectic groups, by ${}^{\mathcal{J}}A$ we denote the matrix multiplication $\mathcal{J}^TAJ$, for any $m_i\times n_n$ matrix A over K. Note that ${}^J(A\,{}^{\mathcal{J}}B) = -B\,{}^{\mathcal{J}}A$.

4. Generators and relations of U associated with root systems of type BC_n

Assume all the notations in the previous sections and let U be any unipotent subgroup defined as in Section 3. By describing generators and relations, we associate U with the positive nonreduced root systems of type BC_n in this section, where obviously $2\leqq n\leqq\frac{l-1}{2}$ for $O(2l,K)$, $2\leqq n\leqq\frac{l}{2}$ for $O(2l+1,K)$, and $2\leqq n\leqq l-1$ for $S_p(2l, K)$.

Referring to [6] and [8], we see that U contains the following elements which are called ***medium*** and ***long root elements*** :

$$X_{R_{i,j}}(A) = X_{i,j}(A) = I+AE_{i,j}-{}^JAE_{-j,-i},$$

$$X_{R_{i,-j}}(A) = X_{i,-j}(A) = \begin{cases} I+AE_{i,-j}-{}^JAE_{j,-i} \text{ if } G = O(K),\\ I+AE_{i,-j}+{}^JAE_{j,-i} \text{ if } G = S_p(2l,K),\end{cases}$$

$$X_{R_{i,-i}}(A) = X_{i,-i}(A) = I + AE_{i,-i}, \quad \text{where} \begin{cases} A = -{}^JA & \text{if } G = O(K), \\ A = +{}^JA & \text{if } G = S_p(2l, K), \end{cases}$$

for any matrices A of suitable sizes, where $1 \leqq i < j \leqq n$, $1 \leqq i \leqq n$.

Furthermore, we see that U contains any element of the following form related to each short root $R = R_{i,n+1}$ which we denote by $X_{R_{i,n+1}}(A, C) = X_{i,n+1}(A, C)$:

$$I + AE_{i,n+1} + A'E_{n+1,-i} + CE_{i,-i},$$

where

(1) $A' = -{}^JA$ and $AA' = C + {}^JC$ if $G = O(K)$,

(2) $A' = {}^JA$ and $AA' = C - {}^JC$ if $G = S_p(2l, K)$,

for an $m_i \times n_n$ matrix A and an $m_i \times m_i$ square matrix C over K. Note that

$$X_{i,n+1}(0, C) = X_{i,-i}(C),$$
$$X_{i,n+1}(A, 0) = I + AE_{i,n+1} + A'E_{n+1,-i},$$

where $AA' = 0$. From now on, whenever we use the notation $X_{i,n+l}(A, C)$, we always assume the above condition either (1) or (2). Also note that

$$X_{i,n+1}(A, C) X_{i,n+1}(B, D) = X_{i,n+1}(A+B, C+D+AB'),$$

whence the set of all $X_{i,n+1}(A, C)$ is a group.

Recall that the ***root subgroup*** of U is defined as the group generated by all the corresponding root elements. For long and medium roots R, we can see the corresponding root subgroup X_R is abelian and just the set of all root elements $X_R(A)$, for

$$X_R(A) X_R(B) = X_R(A+B).$$

But for short roots, we are in a different situation and obtain Proposition 4.4.

In order to factor out the part corresponding to a long root from $X_{i,n+1}(A, C)$, we need the following lemma :

Lemma 4.1. *Let* $X_{i,n+1}(A, C)$ *be defined as above. Then we can take an upper triangular matrix* A^* *with respect to the minor diagonal so that the following equality holds* :

$$X_{i,n+1}(A, C) = X_{i,n+1}(A, A^*) \; X_{i,-i}(C - A^*),$$

where A^* depends only on A. Especially, if AA' is a zero matrix, then we have

$$X_{i,n+1}(A, C) = X_{i,n+1}(A, 0)X_{i,-i}(C).$$

Proof. If $AA' = [b_{s,t}] = \sum b_{s,t}e_{s,t}$, then we take A^* as

$$\sum_{s<-t} b_{s,t}e_{s,t} + \sum_{s=-t} \frac{1}{2} b_{s,t}e_{s,t}.$$

For symplectic groups, note that all the minor diagonal entries $b_{s,-s}$ of AA' are zeroes.

Let $X_{i,n+1}(A, A^*)$ be defined as in the above lemma. Then Lemma 2.1 and Proposition 4.2 enable us to call $X_{i,n+1}(A, A^*)$ the *root element corresponding to a short root* $R_{i,n+1}$ or the *short root element*. Since A^* depends only on A in the above lemma, we denote $X_{i,n+1}(A, A^*)$ simply by $X_{i,n+1}(A)$. And let $Mat(R, K)$ be the set of all matrices A such that each root element $X_R(A)$ makes sense for all positive roots R of type BC_n. Note that each $Mat(R, K)$ is the set of all matrices of suitable sizes with no restriction except for long roots.

Now let's consider the commutator relations $[a ; b] = a^{-1}b^{-1}ab$ of elements a, b in U. All the nontrivial commutator products of elements of U are obtained from the following products related to short roots in addition to those related to medium and long roots in [6] and [8]:

$$[X_{i,n+1}(A,C) ; X_{j,n+1}(B,D)] = \begin{cases} X_{i,-j}(AB') & \text{if } 1 \leqq i < j \leqq n, \\ X_{i,-i}(AB' - BA') & \text{if } 1 \leqq i = j \leqq n, \\ X_{j,-i}(-BA') & \text{if } 1 \leqq j < i \leqq n, \end{cases}$$

$$[X_{i,j}(A) ; X_{j,n+1}(B, D)] = \begin{cases} X_{i,n+1}(AB, A^JD^JA)X_{i,-j}(AD) & \text{if } G = O(K), \\ X_{i,n+1}(AB, -A^JD^JA)X_{i,-j}(AD) & \text{if } G = S_p(2l, K). \end{cases}$$

Since the length of a long root $R_{i,-i}$ is the twice of that of a short root $R_{i,n+1}$, we expect the following proposition 4.2 and corollary 4.3:

Proposition 4.2. *Each long root subgroup of U is generated by the corresponding short root elements.*

Proof. Let $X_{i,-i}(A)$ be any element in the long root subgroup $X_{i,-i}$, where $1 \leqq i \leqq n$. Then, for each $1 \leqq s \leqq m_i$, take B_s as the $m_i \times n_n$ matrix whose last column is the transpose of the s-th row of A with respect to the minor diagonal and other entries are all zeroes, and consider $e_{s,1}$ as the $m_i \times n_n$ matrix whose $(s, 1)$ entry is 1 and other entries are all zeroes so that

$$A = \sum_s e_{s,1}{}^JB_s = \sum_s e_{s,1}{}^JB_s.$$

In the case when $G = O(K)$, since A is an $m_i \times m_i$ square matrix such that $A = -{}^JA$, we have

$$\prod_{1 \leq s \leq m_i} \left[X_{i,n+1}(-\frac{1}{2} e_{s,1}) ; X_{i,n+1}(B_s)\right]$$

$$= X_{i,-i}\left(\frac{1}{2} \sum_s e_{s,1}\, {}^JB_s - \frac{1}{2} \sum_s B_s\, {}^Je_{s,1}\right)$$

$$= X_{i,-i}\left(\frac{1}{2} A - \frac{1}{2}\, {}^JA\right) = X_{i,-i}(A).$$

Similarly, in the case when $G = S_p(2l, K)$, since $A = {}^JA$, we have

$$\prod_{1 \leq s \leq m_i} \left[X_{i,n+1}(\frac{1}{2} e_{s,1}) ; X_{i,n+1}(B_s)\right]$$

$$= X_{i,-i}\left(\frac{1}{2} A + \frac{1}{2}\, {}^JA\right) = X_{i,-i}(A).$$

Therefore, we can generate each long root subgroup by the corresponding short root elements.

Remark. For $O(2l, K)$ and $S_p(2l, K)$, since $n_n \geqq 2$, note that

$$X_{i,n+1}\left(\pm\frac{1}{2} e_{s,1}\right) = X_{i,n+1}\left(\pm\frac{1}{2} e_{s,1}, 0\right)$$
$$X_{i,n+1}(B_s) = X_{i,n+1}(B_s, 0).$$

Corollary 4.3. *Each long root subgroup $X_{i,-i}$ of U is the commutator subgroup of the short root subgroup $X_{i,n+1}$ where $1 \leqq i \leqq n$.*

Therefore, combining Lemma 4.1, Proposition 4.2, and Corollary 4.3, we obtain the following proposition 4.4 :

Proposition 4.4. *Let $X_{i,n+1}$ be any short root subgroup of U, for $1 \leqq i \leqq n$. Then we have*

(1) *$X_{i,n+1}$ is the set of all short root elements $X_{i,n+1}(A)$* modulo $X_{i,-i}$.

(2) *$X_{i,n+1}$ is precisely the set of all elements $X_{i,n+1}(A, C)$.*

(3) *The factor group $X_{i,n+1} / X_{i,-i}$ is abelian.*

Let $A = I + \sum_{s<t} A_{s,t} E_{S,t}$ be any element of U. Then, by commutator relations, we can express A uniquely as the following product of *root elements*:

$$X_{R_{1,2}}(A_{1,2})X_{R_{2,3}}(A_{2,3})\cdots X_{R_{n,n+1}}(A_{n,n+1})X_{R_{1,3}}(B_{1,3})\ X_{R_{2,4}}(B_{2,4})\cdots$$
$$X_{R_{n-1,n+1}}(B_{n-1,n+1})X_{R_{n,-n}}(B_{n,-n})X_{R_{1,4}}(B_{1,4})\cdots X_{R_{1,-1}}(B_{1,-1}),$$

where $R_{s,t}$ are all the positive roots in the nonreduced root system of type BC_n arranged in increasing order, and $B_{s,t} \in Mat(R_{s,t}, K)$ are chosen properly. Furthermore, we have the following theorem

Theorem 4.5. *Let U be a unipotent subgroup of orthogonal and symplectic groups G defined as in Section 3. Then U is generated by the root elements $X_R(A)$ corresponding to all fundamental roots R in the nonreduced root system of type BC_n for $A \in Mat(R, K)$.*

Proof. By the above remark, it is enough to generate all root elements of U. Referring to [6] and [8], we can generate all medium root elements by fundamental root elements. Owing to Proposition 4.2, we only need to generate root elements corresponding to short roots $R_{i,n+1}$ for $1 \leqq i \leqq n-1$. Suppose $X_{i,n+1}(A) = X_{i,n+1}(A, A^*)$ is any short root element of U and let $A_s \in Mat(R_{i,n+1}, K)$ be the matrix whose s-th row is the same as that of A and other entries are all zeroes, and let $B_s \in Mat(R_{i+1,n+1}, K)$ be the matrix whose first row is the same as that of A and other entries are all zeroes, and consider $e_{s,1} \in Mat(R_{i,i+1}, K)$ as usual. Then we can generate each $X_{i,n+1}\ (A_s, A_s^*)$ case by case as follows:

(1) Case when $G = S_p(2l, K)$, $l \geqq 3$: We have

$$X_{i,n+1}(A_s, A_s^*) = X_{i,n+1}(A_s, 0) = [X_{i,i+1}(e_{s,1})\ ;\ X_{i+1,n+1}(B_s, 0)].$$

(2) Case when $G = O(2l+1, K)$, $l \geqq 4$: Since n_n is an odd number, put $n_n = 2y+1$. If $dim V_n < l$, then $n_n \geqq 3$. So we have the following:

$$A = \sum_{1 \leqq s \leqq n_i} A_s \quad \text{and} \quad A_s = A_s^1 + A_s^2 + A_s^3,$$

where $A_s^k \in Mat(R_{i,n+1}, K)$ for $1 \leqq k \leqq 3$, and

$$A_s^1 = \sum_{1 \leqq t \leqq y} a_{s,t} e_{s,t}, \quad A_s^2 = \sum_{-y \leqq t \leqq -1} a_{s,t} e_{s,t}, \quad \text{and } A_s^3 = a_{s,y+1} e_{s,y+1}.$$

Similarly, consider B_s as $B_s^1 + B_s^2 + B_s^3$. Then we have

$$\prod_{1\leq k\leq 3}[X_{i,i+1}(e_{s,1})\,;\,X_{i+1,n+1}(B_s^k\ B_s^{k^*})]\ X_{i,-(i+1)}(-e_{s,1}\ B_s^{k^*})$$
$$= X_{i,n+1}(A_s^1, 0)\ X_{i,n+1}(A_s^2, 0)X_{i,n+1}(A_s^3, -\frac{1}{2}(a_{s,y+1})^2\ e_{s,-s})$$
$$= X_{i,n+1}(A_s, -A_s^1\ {}^J(A_s^2)-\frac{1}{2}(a_{s,y+1})^2\ e_{s,-s}) = X_{i,n+1}(A_s, A_s^*).$$

In the above equalities, note that $A_s^{k^*}=B_s^{k^*}=0$ for $k = 1,2$ and that $(A_s^1+A_s^2)\ {}^J(A_s^3) = 0$.

If $dim V_n = l$, then $n_n = 2y+1 = 1$, whence $y = 0$. Thus, considering A_s just as a column matrix A_s^3, we follow the above case of $n_n \geqq 3$.

(3) Case when $G = O(2l, K)$, $l \geqq 5$: Since $n_n = 2y \geqq 2$, we consider $A_s = A_s^1 + A_s^2$ and follow the above case (2).

Then, for all the cases, we have

$$X_{i,n+1}(A, A^*) = \prod_{1\leq s\leq m_i} X_{i,n+1}(A_s, A_s^*).$$

Thus, by inductive arguments on i any short root element $X_{i,n+1}(A)$ is generated by root elements corresponding to fundamental roots, for $1 \leqq i \leqq n-1$. Thus the proof is done.

Therefore, the positive irreducible nonreduced root systems of type BC_n can be associated with some nonmaximal unipotent subgroups U of orthogonal and symplectic groups over a field of characteristic not equal to 2.

Remark. For $O(2l, K)$ and $S_p(2l, K)$, we note that root subgroups $X_{i,-i}$ and $X_{i,n+1}$ are generated by root elements of the form

$$X_{i,n+1}(A) = X_{i,n+1}(A, A^*) = X_{i,n+1}(A, 0) = I+AE_{i,n+1}-A'E_{n+1,-i},$$

where $AA' = 0$.

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